Numerical Holography

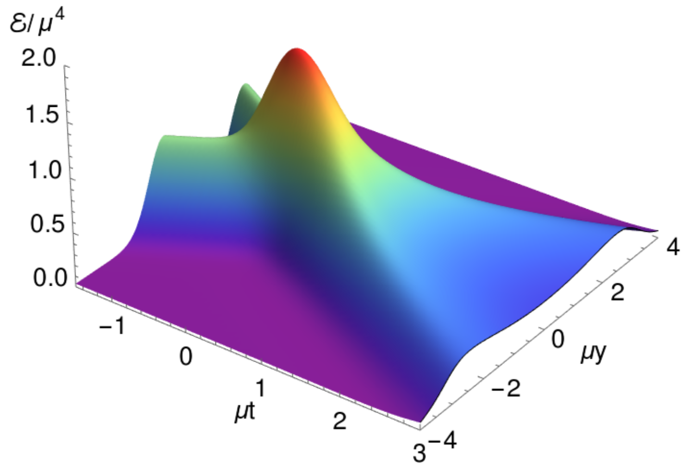
Christian Ecker

Utrecht University

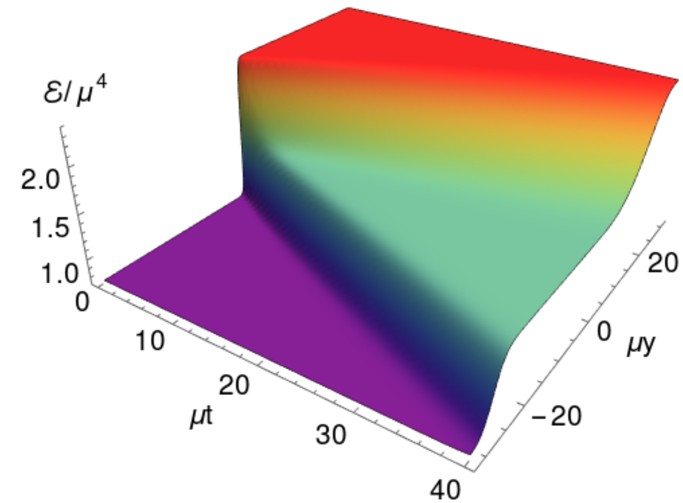
TIFR ICTS School
Bangalore, India, April 12, 2019

Myriad colorful ways of AdS/CFT

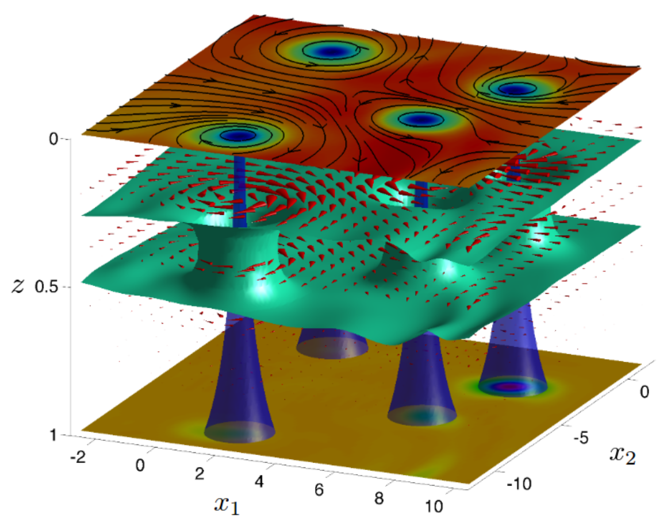
Shock Wave Collisions



Steady State Formation

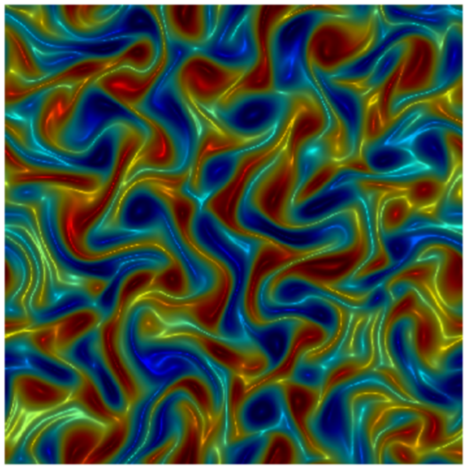


Superfluids



[Chesler, Yaffe, 1212.0281]

Turbulence



[Adams, Chesler, Yaffe, 1307.7267]

Selection of Useful References

- Introductory Book on AdS/CFT:

  Ammon, Erdmenger, Cambridge University Press (2015)

- Book on AdS/CFT and Heavy Ion Collisions:

  Casalderray-Solana, Liu, Mateos, Rajagopal, Wiedemann, Cambridge University Press (2014)

- Review on Characteristic Method for Numerical Holography:

  Chesler, Yaffe (1309.1439)

- Book on Spectral Methods:

  Boyd, Dover Publications (2001)

Outline

- **Part I: Introduction**

  AdS/CFT Correspondence, Fefferman-Graham Expansion, Holographic Renormalization, ...

- **Part II: Numerical GR on AdS**

  Characteristic Formulation, Homogeneous Isotropization, Spectral Methods, Boost Invariant Hydrodynamization, ...

- **Part III: Advanced Examples**

  Shock Wave Collisions, Correlations, Entanglement Entropy, QNEC, Steady State Formation

- **Summary**

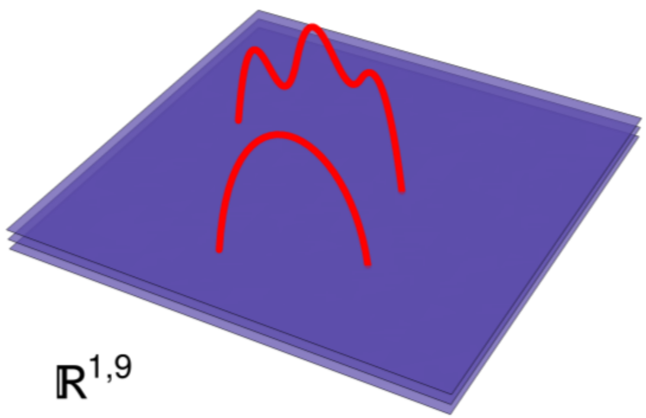
Part I: Introduction

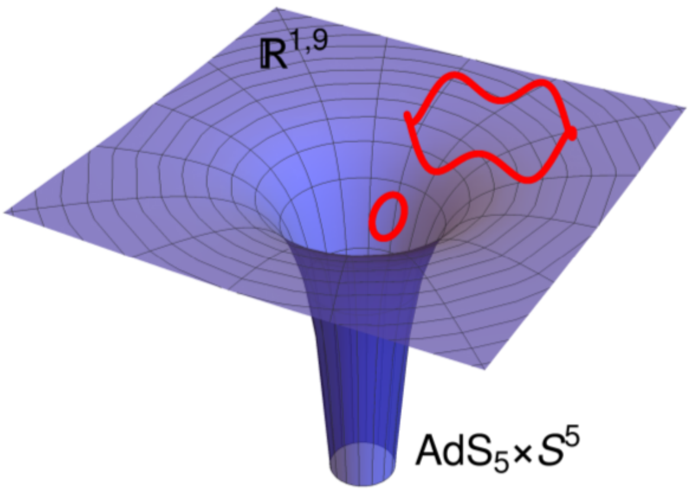
AdS/CFT Correspondence

- Duality between SU(N) $\mathcal{N}$=4 SYM theory and type IIB string theory on $AdS_5 \times S^5$.

[Maldacena hep-th/9711200]

$$\langle e^{\int d^d x \mathcal{O}(x)\phi_{(0)}(x)} \rangle_{CFT} = \mathcal{Z}_{string}[\Phi(r,x)]$$





- Correspondence relates parameters of the two theories

$$g_{YM}^2 = 2\pi g_s \qquad 2g_{YM}^2 N = L^4/l_s^4$$

Supergravity Limit

- Assuming point like strings ($l_s \to 0$) and small string coupling ($g_s \ll 1$) reduces string theory to classical (super)gravity.

$$\mathcal{Z}_{string} \approx e^{S_{ren}[\Phi_c]}$$

- Corresponds to large N and large 't Hooft coupling limit on the field theory side.

$$N \to \infty \qquad\qquad \lambda = 2g_{YM}^2 N \to \infty$$

- Obtain observables in strongly coupled QFT, by doing classical gravity calculations.
- Correlation functions on field theory side are obtained from

$$\langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \rangle_{CFT} = \frac{\delta^n S_{ren}[\Phi_c]}{\delta\phi(x_1) \ldots \delta\phi(x_n)}\Big|_{\phi=0}$$

Holographic Dictionary

- Every bulk field has a corresponding operator in the boundary theory

Gravity side		Gauge theory side	
metric	$g_{\mu\nu}$	$T^{\mu\nu}$	stress tensor
scalar field	ϕ	$\mathcal{O}$	scalar operator
gauge field	A_μ	J^μ	global sym. current
...			

- Geometry in the bulk corresponds to state in the field theory.
- Hawking temperature of BH corresponds to temperature of the field theory.
- In this lecture we restrict to states which are dual to black brane solutions of

$$S = -\frac{1}{16\pi G_N}\int d^{d+1}x\sqrt{g}\left(R + \frac{d(d-1)}{L^2}\right) - \frac{1}{8\pi G_N}\int d^d x\sqrt{\gamma}K$$

- Need explicit relation between bulk metric and field theory stress tensor.

Fefferman-Graham Expansion

- The metric can be expanded in the radial coordinate

$$ds^2 = G_{MN} dx^M dx^N = L^2 \left(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{\mu\nu}(\rho, x) dx^\mu dx^\nu \right)$$

- If G_{MN} satisfies Einstein equations, then $g_{\mu\nu}$ has the following expansion

$$g_{\mu\nu}(\rho, x) = g_{(0)\mu\nu}(x) + \rho g_{(2)\mu\nu}(x) + \ldots + \rho^{d/2} \left(\log(\rho) h_{(d)\mu\nu}(x) + g_{(d)\mu\nu}(x) \right) + \ldots$$

- Solving Einstein equations order by order to express coefficients

$$g_{(2)\mu\nu}(x) = \frac{L}{d-2} \left(R_{(0)\mu\nu} - \frac{1}{2(d-1)} R_{(0)} g_{(0)\mu\nu} \right), \ldots$$

- Important remarks:

  Logarithms are related to conformal anomaly, only present when d is even.

  Coefficients from order d on can only be extracted from full bulk solution.

Holographic Renormalization

- Putting the asymptotic expansion into the action and evaluate at cutoff gives

$$S_\epsilon = -\frac{1}{16\pi G_N}\int d^dx\sqrt{g_{(0)}}\left(a_{(0)}\epsilon^{-d/2} + a_{(2)}\epsilon^{-d/2+1} + \ldots - \log a_{(d)}\epsilon\right) + S_{finite}$$

- To renormalize the action we have to add appropriate counter terms (ambiguities!)

$$S_{ren} = \lim_{\epsilon\to 0}(S_\epsilon + S_{ct})$$

- Varying the renormalized action w.r.t. the boundary metric gives the stress tensor

$$\langle T_{\mu\nu}(x)\rangle = -\frac{2}{\sqrt{g_{(0)}}}\frac{\delta S_{ren}}{\delta g^{\mu\nu}_{(0)}(x)}$$

- Once we have the metric we can extract the field theory stress tensor

$$\langle T_{\mu\nu}\rangle = \frac{4}{16\pi G_N}\left(g_{(4)\mu\nu} + \frac{1}{8}\left(\mathrm{Tr}g^2_{(2)} - (\mathrm{Tr}g_{(2)})^2\right)g_{(0)\mu\nu} - \frac{1}{2}(g^2_{(2)})_{\mu\nu} + \frac{1}{4}g_{(2)\mu\nu}\mathrm{Tr}g_{(2)}\right)$$

[de Haro, Skenderis, Solodukhin, hep-th/0002230]

Holographic Thermalization

- Use black hole formation as model for thermalization/hydrodynamization.
- Relaxation and equilibration of the black hole corresponds to equilibration and thermalization of the field theory state.
- Strategy: prepare far-from equilibrium state and follow time evolution
- Two ways of preparing excited states on the gravity side:

  1) Start with non-equilibrium bulk geometry

  2) Start with equilibrium bulk geometry and turn on boundary source (quench in boundary theory)

Part II: Numerical GR on AdS

Characteristic Formulation

- AdS is not globally hyperbolic, i.e. has no Cauchy slice.

  Always need IC'+BC's to obtain well defined initial value problem.

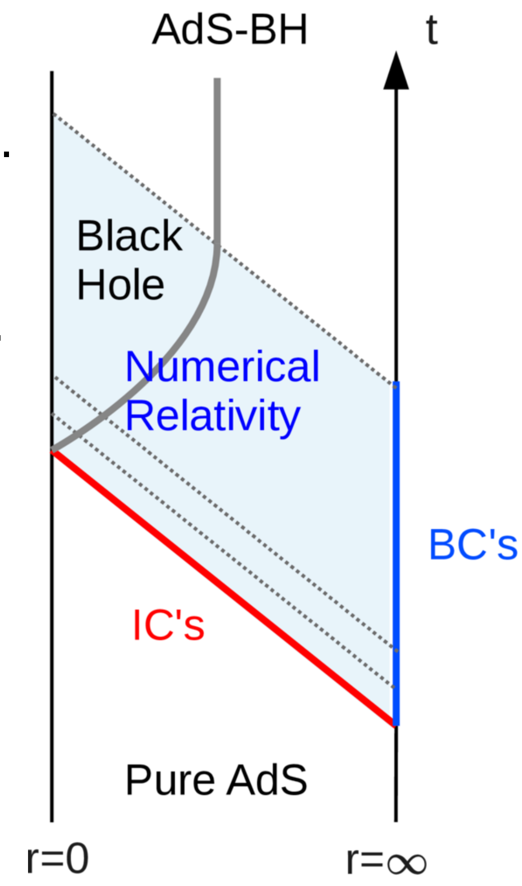
- Using light-like slicing, results in characteristic formulation of GR.

  Realized by generalized Eddington-Finkelstein coordinates

$$ds^2 = 2dvdr + \frac{r^2}{L^2} g_{\mu\nu}(r, x^\mu) dx^\mu dx^\nu$$

- Has residual gauge freedom which can be used to fix position of the apparent horizon.

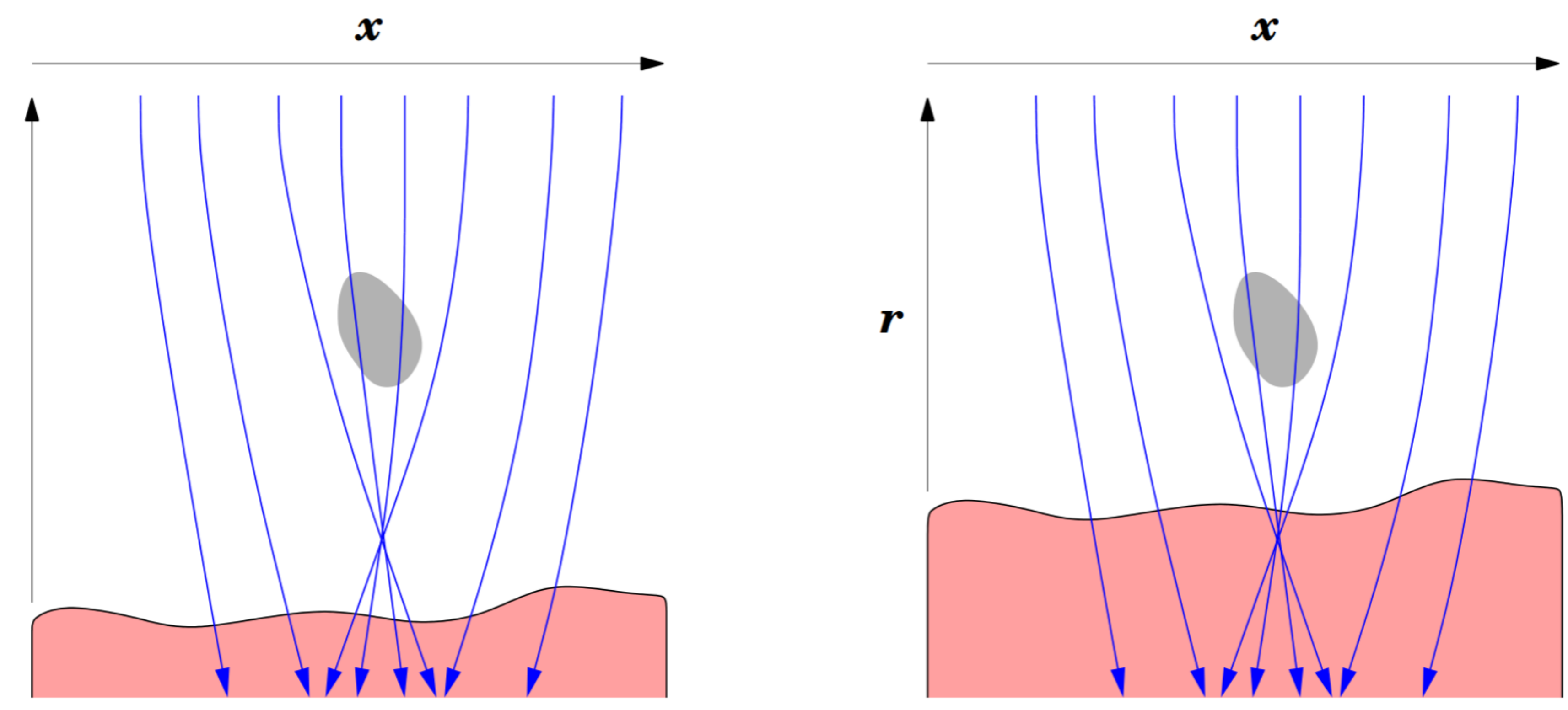
$$r \to \bar{r} \equiv r + \xi(x^\mu)$$



Caustics

- Penrose focusing theorem: *“Matter focuses light.”*

  Light like geodesics can form caustics which destroy the coordinate system.
- Increase regulator energy density to hide caustics behind horizon.



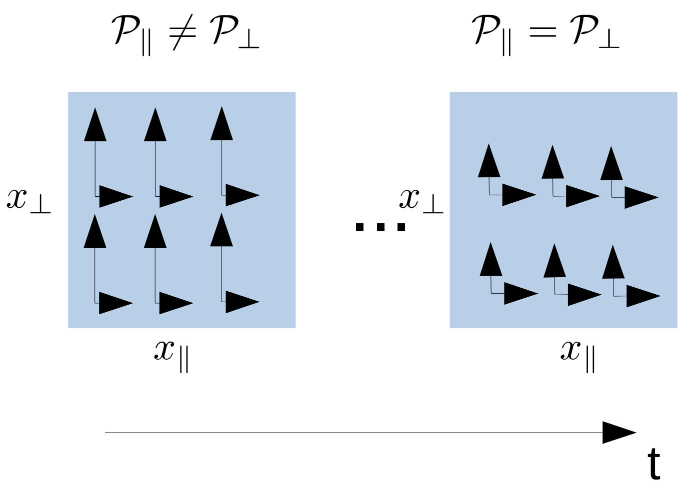
[picture: Chesler, Yaffe,1309.1439]

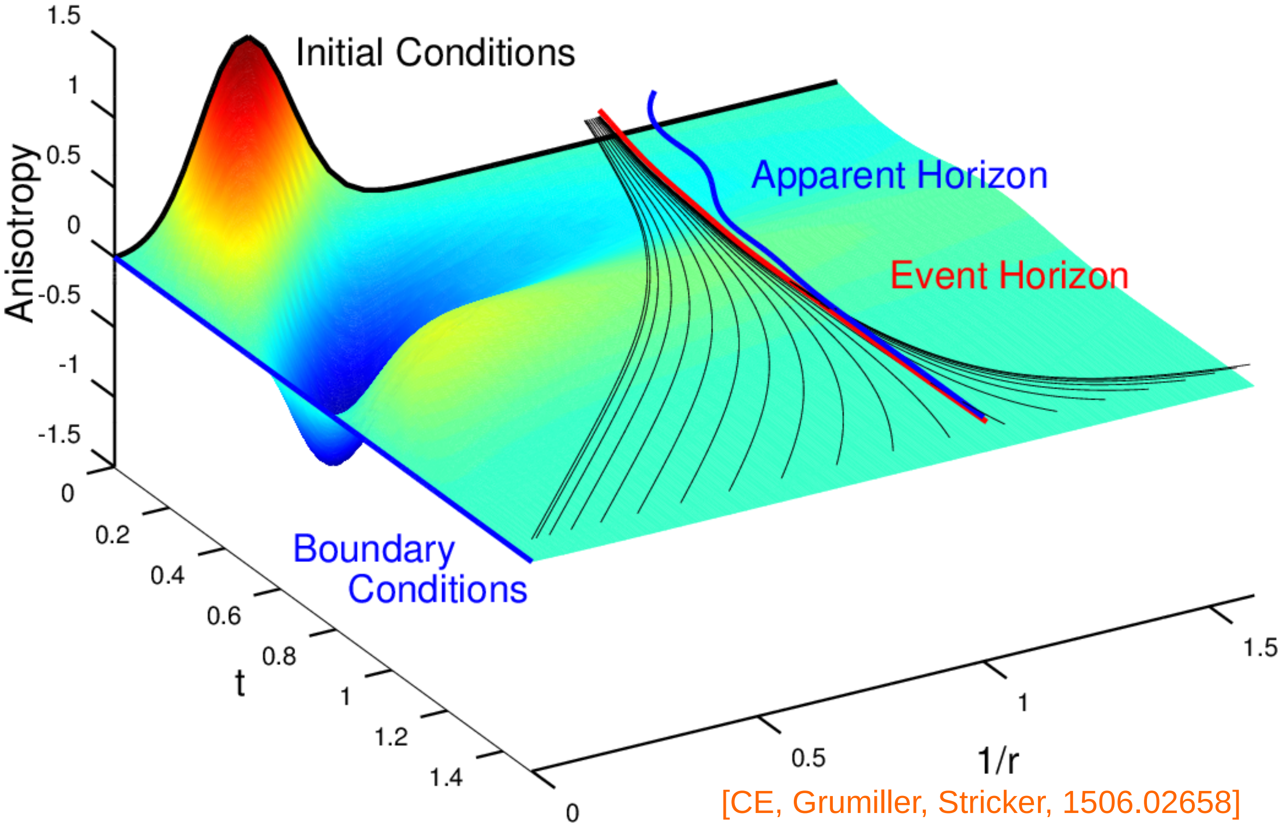
Homogeneous Isotropization

- Homogeneous, initially anisotropic plasma relaxes to its isotropic equilibrium state.

[Chesler, Yaffe, 0812.2053]

$$ds^2 = -A(r,v)dv^2 + 2dvdr + S(r,v)^2(e^{-2B(r,v)}dx_\|^2 + e^{B(r,v)}d\vec{x}_\perp^2)\,.$$





[CE, Grumiller, Stricker, 1506.02658]

Characteristic bulk equations

- For this example we need to solve the 5D Einstein equations

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \qquad \text{s.t.} \qquad ds^2|_{r\to\infty} = r^2(-dt^2 + d\vec{x}^2)$$

- Derivatives along ingoing (prime) and outgoing (dot) null geodesics

$$h' \equiv \partial_r h \qquad \dot{h} \equiv \partial_v h + \tfrac{1}{2} A \partial_r h$$

- Einstein equations decouple into a nested set of ODEs

IC's: $B_{v=v_0} \longrightarrow$

$$S'' + \tfrac{1}{2} B'^2 S = 0 \quad (1)$$

$$S(\dot{S})' + 2S'\dot{S} - 2S^2 = 0 \quad (2)$$

$$S(\dot{B})' + \tfrac{3}{2}(S'\dot{B} + 2B'\dot{S}) = 0 \quad (3)$$

$$A'' + 3B'\dot{B} - 12S'\dot{S}/S^2 + 4 = 0 \quad (4)$$

$$\ddot{S} + \tfrac{1}{2}(\dot{B}^2 S - A'\dot{S}) = 0 \quad (5)$$

$\xrightarrow{A}$ $\dot{B} = \partial_v B + \tfrac{1}{2} A \partial_r B$ $\xrightarrow{\partial_v B}$ $B_{(v+\Delta v)} = B_{(v)} + \Delta v \partial_v B_{(v)}$ $\longrightarrow$ (1)

- Use constraint (1) to prepare IC's, constraint (5) to monitor accuracy.

Near boundary analysis

- Near the boundary the Einstein equations can be solved with power series

$$A(r,v) = r^2 + \frac{a_4(v)}{r^2} + \frac{a_4'(v)}{2r^3} + \mathcal{O}(r^{-4}) \,,$$
$$S(r,v) = r - \frac{a_4'(v)}{20r^4} - \frac{b_4(v)^2}{7r^7} + \mathcal{O}(r^{-8}) \,,$$
$$B(r,v) = \frac{b_4(v)}{r^4} + \frac{b_4'(v)}{r^5} + \mathcal{O}(r^{-6})$$

- $a_0(v) = 1, s_0(v) = 1, b_0(v) = 0$ are fixed by BC's: $ds^2|_{r\to\infty} = r^2(-dt^2 + d\vec{x}^2)$
- Coefficient $b_4(v)$ remains undetermined and needs to be extracted from numerics.
- $b_4(v)$ and $a_4(v)$ contain information on the field theory stress tensor

$$\langle T^{\mu\nu}\rangle = \frac{N_c^2}{2\pi^2}\mathrm{diag}\left(\mathcal{E}, \mathcal{P}(t), \mathcal{P}_\perp(t), \mathcal{P}_\perp(t)\right)$$

$$\mathcal{E} = -\frac{3}{4}a_4 \,, \qquad P_{||}(t) = -\frac{1}{4}a_4 - 2b_4(t) \,, \qquad P_\perp(t) = -\frac{1}{4}a_4 + b_4(t)$$

- At 5th order one recovers energy conservation: $a_4'(v) = 0 \rightarrow a_4(v) = \mathrm{const.}$

Field Redefinitions

- Use inverse radial coordinate $z = \frac{1}{r}$ in which the AdS boundary is at $z = 0$.
- No need for cutoff, can factor out divergent parts of near boundary expansion

$$A(z,v) \to \frac{1}{z^2} + z\tilde{A}(z,v)\,, \quad S(z,v) \to \frac{1}{z} + z^2\tilde{S}(z,v)\,, \quad B(z,v) \to z^3\tilde{B}(z,v)\,,$$

- Redefined fields are tailor made for reading off the stress tensor

$$b_4(t) = \tilde{B}'(0,t)\,, \quad a_4 = \tilde{A}'(0,t)\,,$$

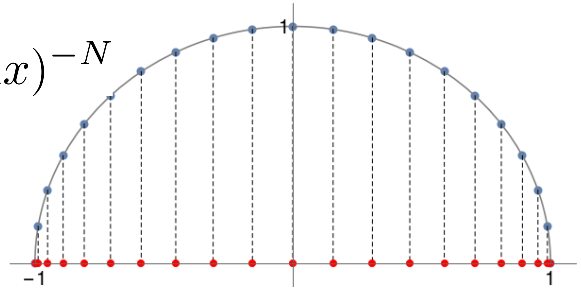
- On each slice we have to solve boundary value problems (BVP) with BCs fixed by near boundary expansion, e.g. equation (4) takes the form

$$\tilde{A}'' + \frac{4}{z}\tilde{A}' + \frac{2}{z^2}\tilde{A} = j_A\,, \quad BCs: \tilde{A}(0,v_0) = 0\,, \tilde{A}'(0,v_0) = a_4$$

Spectral Method

- Expand in terms of Chebyshev polynomials

$$y(x) \approx \sum_{i=0}^{N-1} c_i T_i(x)\,, \quad T_i(cos(x)) = cos(ix)$$



- Highly efficient because of spectral convergence: $error \sim (\Delta x)^{-N}$
- Spectral (Chebyshev) grid

$$x_i = \cos(i\pi/N) \qquad i = 0, \ldots, N$$

- Differentiation on the spectral grid is obtained by matrix multiplication

$$y_i := y(x_i) \qquad y_i' = D_{ij} y_j\,, \quad y_i'' = D_{ij}^2 y_j\,, \ldots \qquad \int dx y_i = D_{ij}^{-1} y_j$$

- Spectral matrix

$$D_{00} = \frac{2N^2+1}{6}\,, \qquad D_{NN} = -\frac{2N^2+1}{6} \quad D_{jj} = \frac{-x_j}{2(1-x_j^2)} \qquad j = 1, \ldots, N-1\,,$$

$$D_{ij} = \frac{c_i}{c_j} \frac{(-1)^{i+j}}{(x_i - x_j)} \qquad j \neq j \quad i, j = 0, \ldots, N\,, \quad c_0 = c_N = 2\,, c_i = 1$$

Boundary Value Problem

- Example of simple boundary value problem (has analytic solution to compare with)

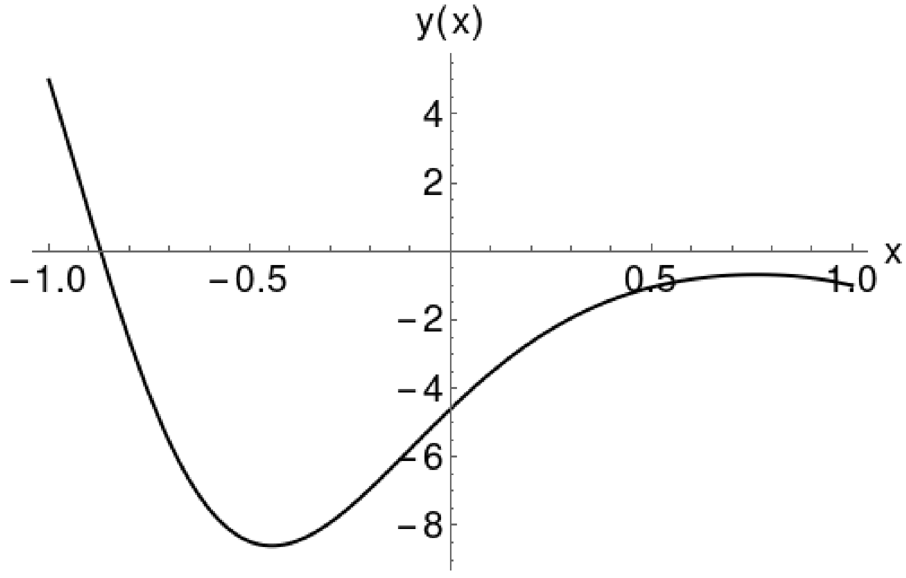
$$y''(x) + y'(x) - 20xy(x) = 0\,, \qquad \mathrm{BC:} \quad y(-1) = 5\,, y(1) = -1$$

- Express differential operator matrix and source vector

$$L = \partial_x^2 + \partial_x - 20x \to L_{ij} = D_{ij}^2 + D_{ij} - 20\mathrm{diag}(x)_{ij}$$

$$S = 0 \to S_i = (0, 0, 0, \dots, 0, 0, 0)$$

- Solution vector y_i is obtained by (dense) matrix inversion $y_i = L_{ij}^{-1} S_j$



Boundary Value Problem

- Example of simple boundary value problem (has analytic solution to compare with)

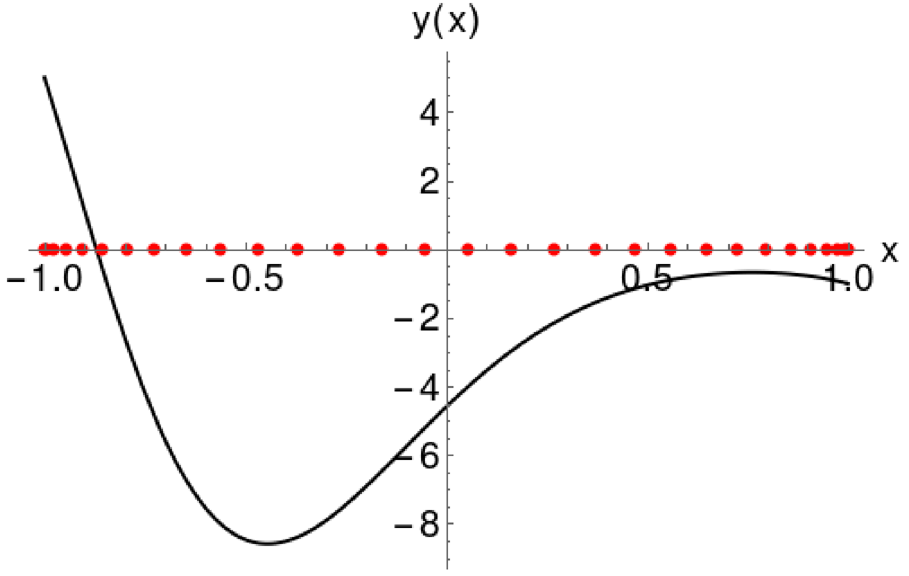
$$y''(x) + y'(x) - 20xy(x) = 0\,, \qquad \mathrm{BC:} \quad y(-1) = 5\,, y(1) = -1$$

- Express differential operator matrix and source vector

$$L = \partial_x^2 + \partial_x - 20x \to L_{ij} = D_{ij}^2 + D_{ij} - 20\mathrm{diag}(x)_{ij}$$

$$S = 0 \to S_i = (0, 0, 0, \ldots, 0, 0, 0)$$

- Solution vector y_i is obtained by (dense) matrix inversion $y_i = L_{ij}^{-1} S_j$

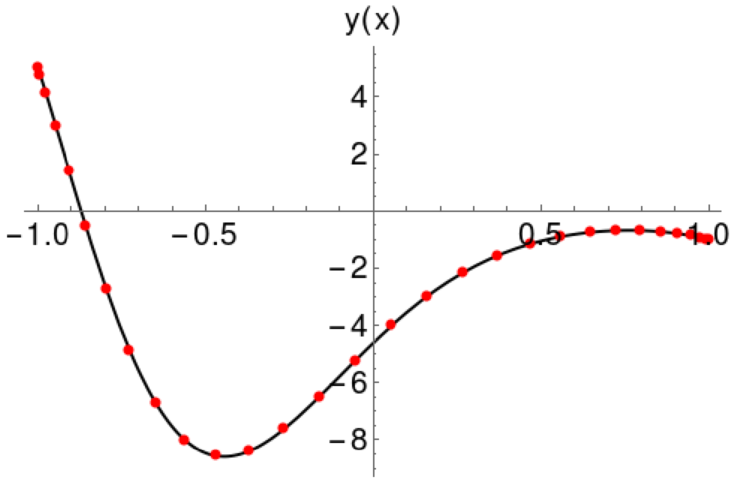


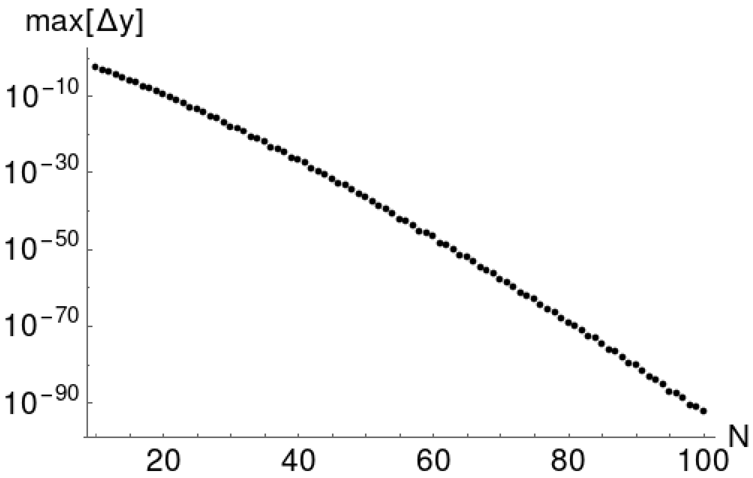
Boundary Conditions

- Differential equation needs boundary conditions! $y(-1) = y_1 = 5\,, y(1) = y_N = -1$
- 1) Boundary bordering:

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ L_{21} & L_{22} & L_{23} & \dots & L_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ L_{N-1,1} & L_{N-1,2} & L_{N-1,3} & \dots & L_{N-1,N} \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-1} \\ y_N \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}$$

- 2) Or basis recombination: use basis functions which explicitly satisfy numerical BCs





Time Stepping

- Simplest way: 1st order Euler method

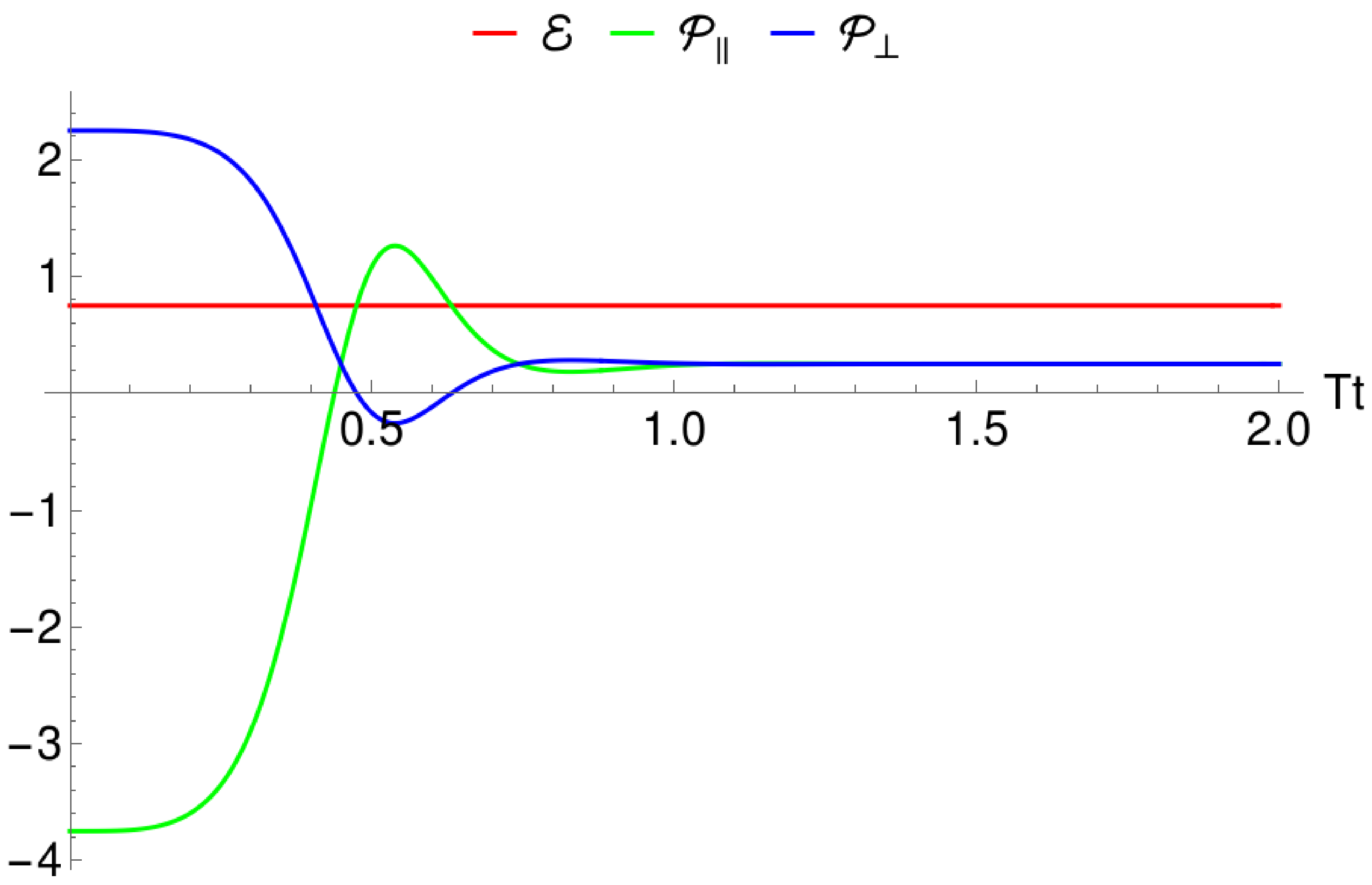
$$y_i(t+\Delta t) = y_i(t) + \Delta t y_i'(t)$$

- More accurate: 4th order Adams-Bashforth

$$y_i(t+4\Delta t) = y_i(t+3\Delta t) + \frac{\Delta t}{24}\left(55y_i'(t+3\Delta t) - 59y_i'(t+2\Delta t) + 37y_i'(t+\Delta t) - 9y_i'(t)\right)$$

- First few steps we have to use lower order scheme

Energy Density and Pressure



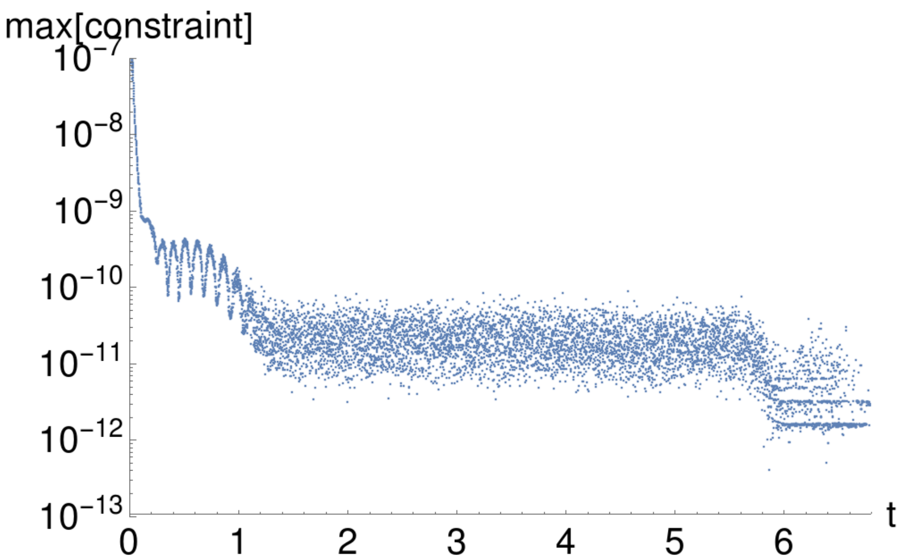
Quasinormal Modes

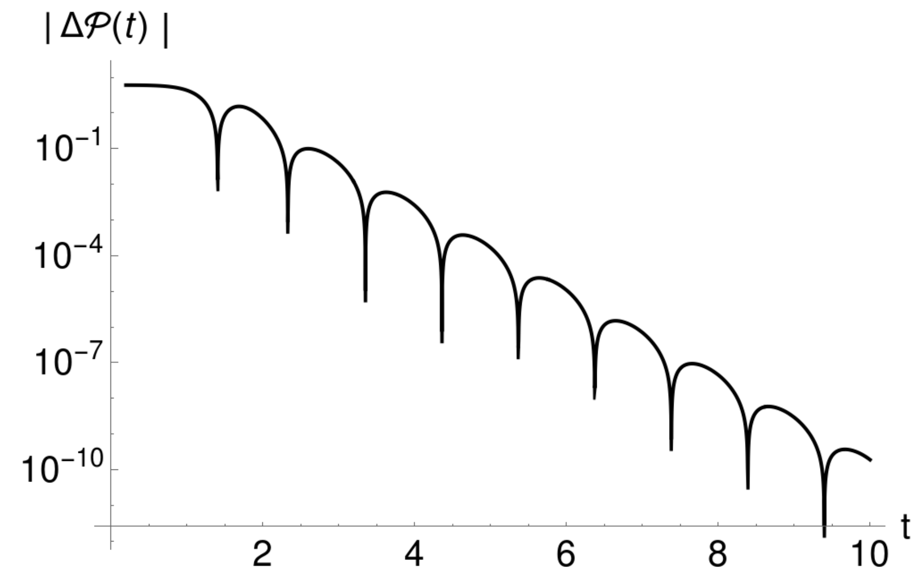
- Late time dynamics is described by quasi normal modes

$$\Delta\mathcal{P} \propto \mathrm{Re}\sum_n c_n e^{-\lambda_n t}\,, \quad \frac{\lambda_1}{\pi T} = 2.746676 + 3.119452i\,, \quad \frac{\lambda_2}{\pi T} = 4.763570 + 5.169521i\,, \ldots$$

[Starinets, hep-th/0207133]

- Accuracy with only 30 grid points is good enough to extract lowest QNM





Boost Invariant Hydrodynamization

- Use proper time and rapidity coordinates and assume independence on y.

$$t = \tau \cosh y \,, \quad x_{\|} = \tau \sinh y \qquad ds^2 = -d\tau^2 + \tau^2 y^2 + d\vec{x}_{\perp}^2$$

$$ds^2 = -A(r,\tau)d\tau^2 + 2d\tau dr + S(r,\tau)^2(e^{-2B(r,\tau)}dx_{\|}^2 + e^{B(r,\tau)}d\vec{x}_{\perp}^2) \,.$$

[Chesler, Yaffe, 0906.4426]

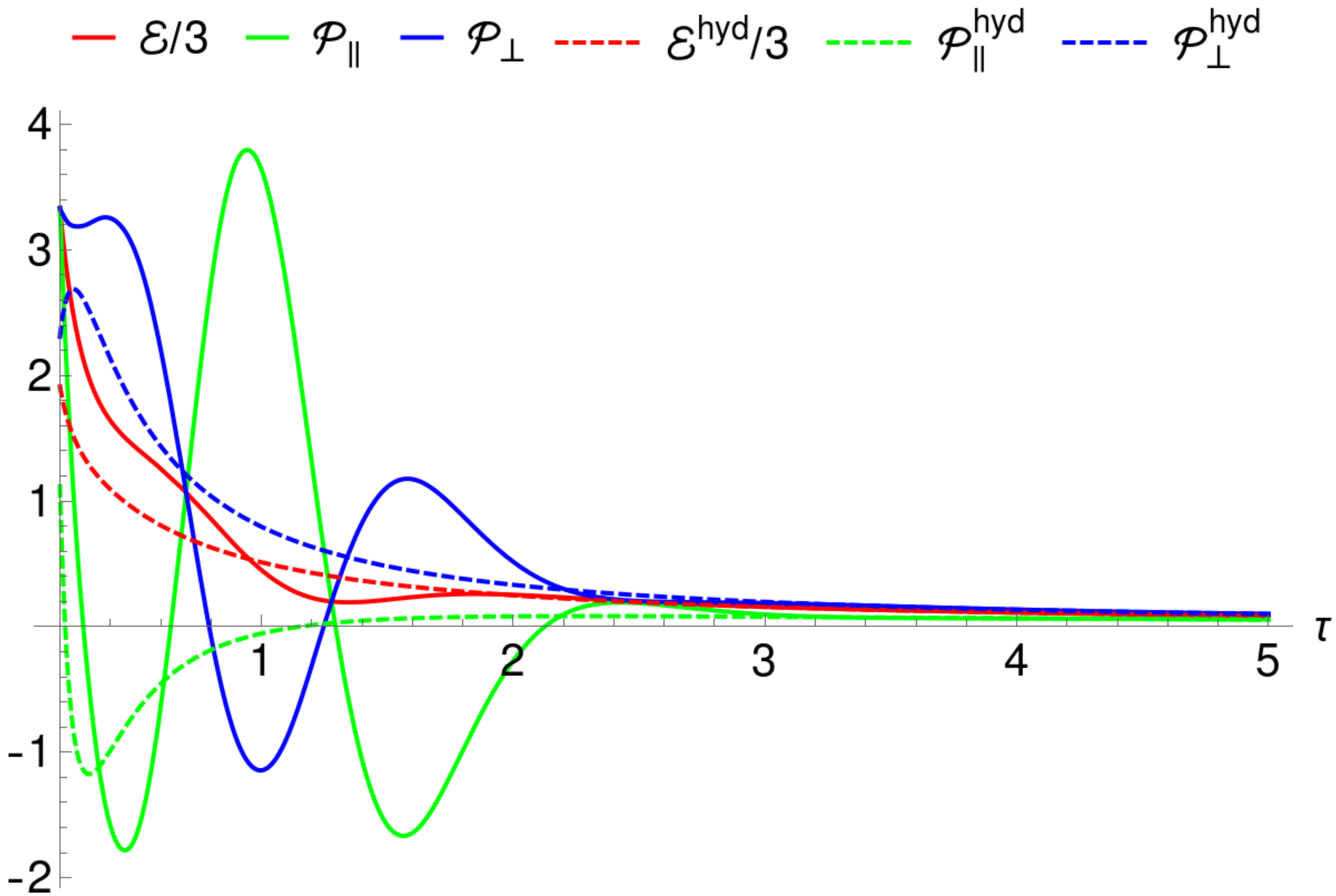
- Late time: 2nd order hydrodynamic expansion of boost invariantly expanding conformal fluid

$$\begin{aligned}
\mathcal{E}(\tau) &= \frac{3\pi^4\Lambda^4}{4(\Lambda\tau)^{4/3}}\left(1 - \frac{2c_1}{(\Lambda\tau)^{2/3}} + \frac{c_2}{(\Lambda\tau)^{4/3}} + \ldots\right) , \\
\mathcal{P}_{\perp}(\tau) &= \frac{\pi^4\Lambda^4}{4(\Lambda\tau)^{4/3}}\left(1 - \frac{c_2}{(3\Lambda\tau)^{4/3}} + \ldots\right) , \\
\mathcal{P}_{\|}(\tau) &= \frac{\pi^4\Lambda^4}{4(\Lambda\tau)^{4/3}}\left(1 - \frac{2c_1}{(\Lambda\tau)^{2/3}} + \frac{5c_2}{(3\Lambda\tau)^{4/3}} + \ldots\right) ,
\end{aligned}$$

$$\mathcal{N} = 4\,\mathrm{SYM}: \quad c_1 = \frac{1}{3\pi} \,, \quad c_2 = \frac{1 + 2\ln 2}{18\pi^2}$$

[Baier,Romatschke, Son, Starinets, Stephanov, 0712.2451]

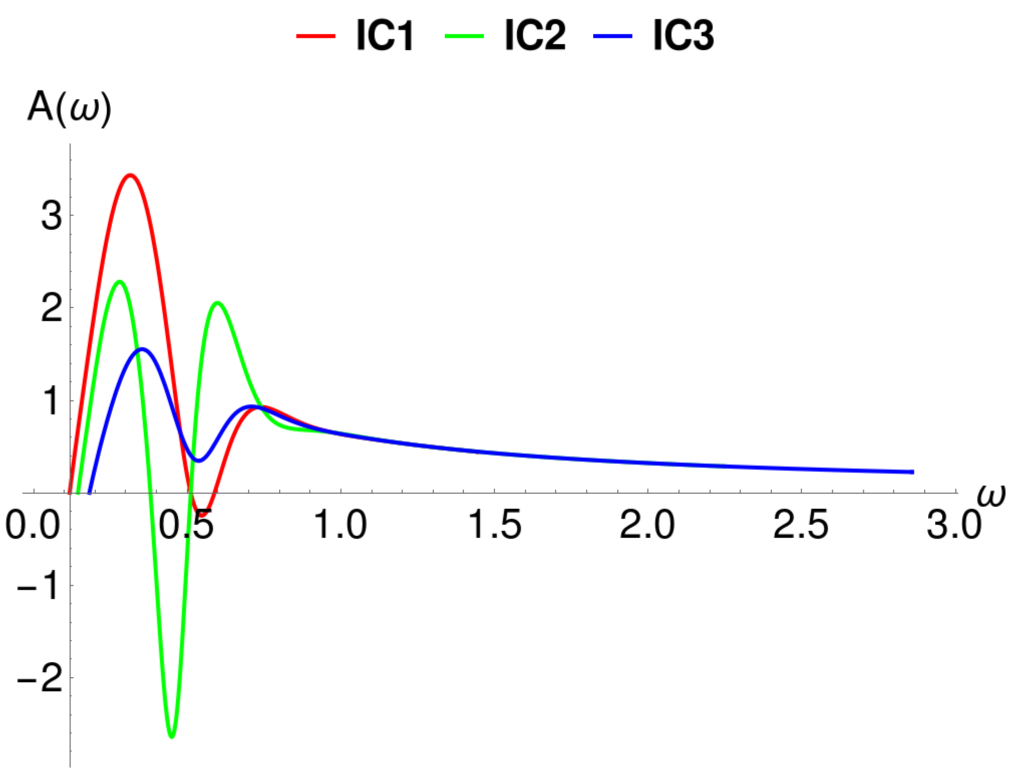
Results for Stress Tensor



Universality

- Pressure anisotropy $A(\omega)$ shows universal behavior when expressed in terms of dimensionless variable ω [Jankowski, Plewa, Spalinski, 1411.1969]

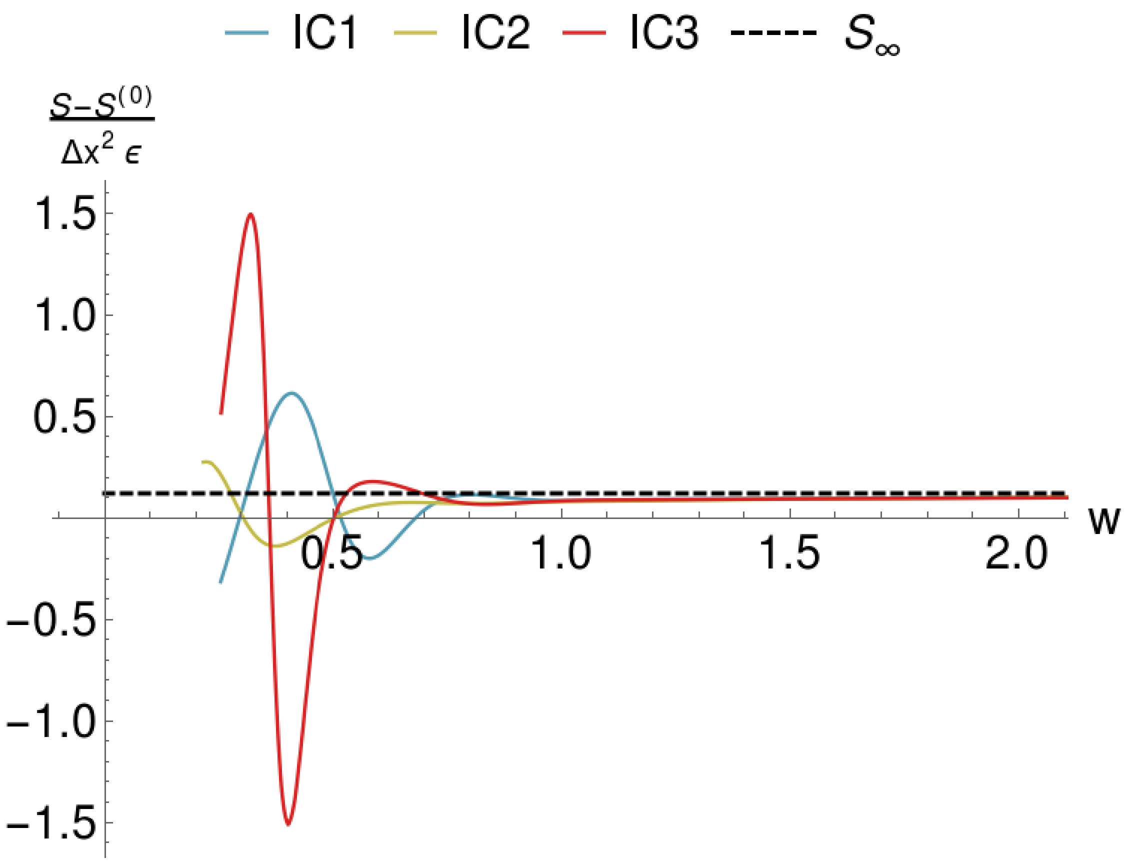
$$A(\omega) = \frac{\mathcal{P}_\perp(\omega) - \mathcal{P}_{||}(\omega)}{\mathcal{P}(\omega)}\,, \quad \mathcal{P} = \mathcal{E}/3\,, \quad \omega = \tau T(\tau)$$



Non-Local Universality

- Non-local observables like two-point functions and entanglement entropy can also brought to universal form

[CE, Jankowski, Spalinski, to appear]

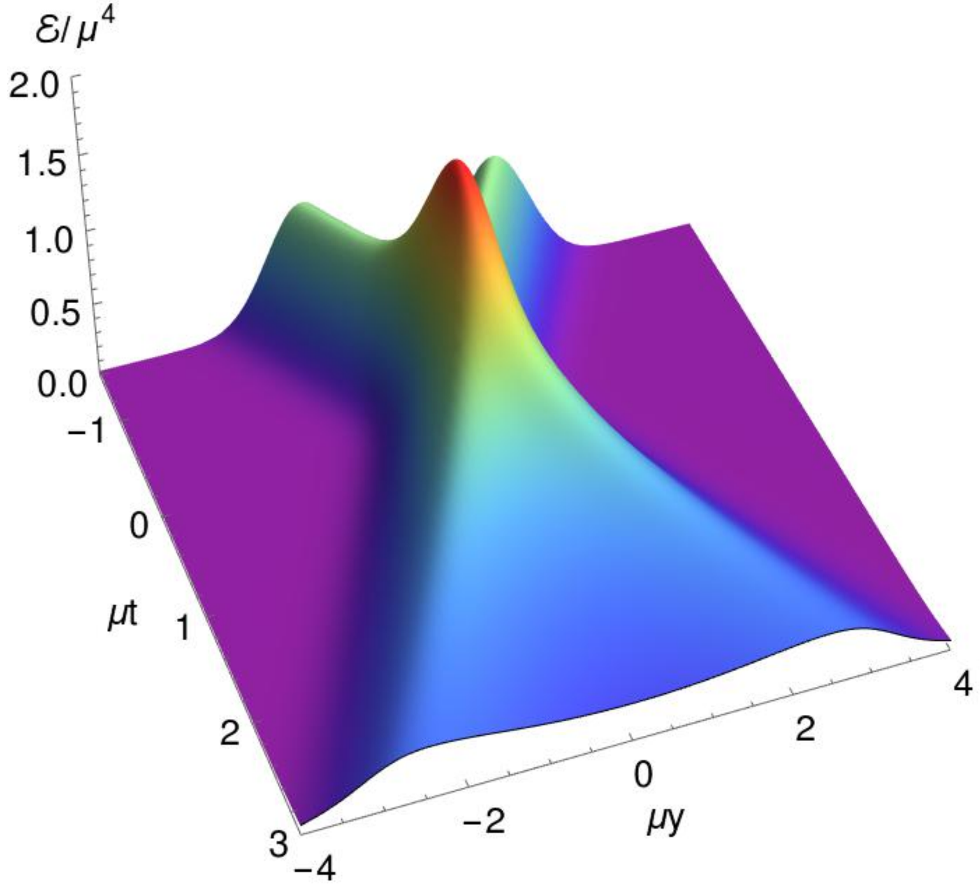


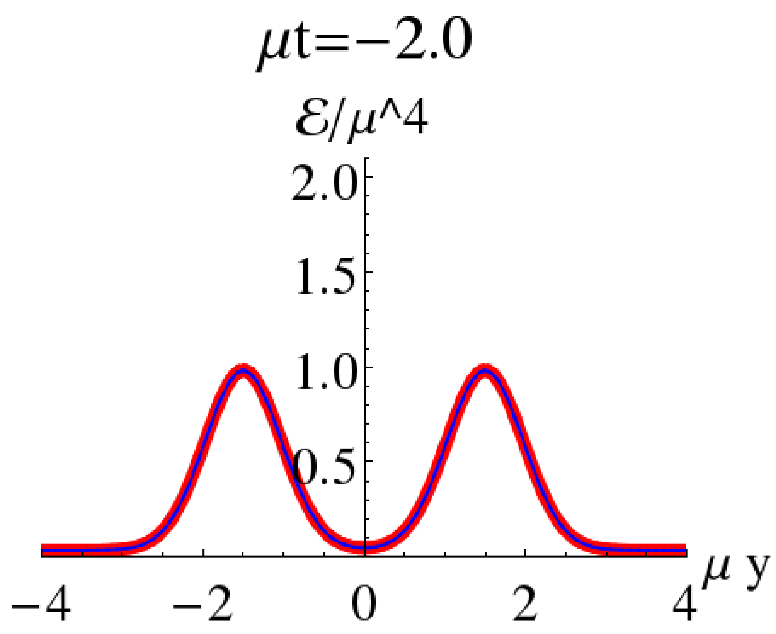
Part III: Advanced Topics

Holographic Shock Wave Collisions

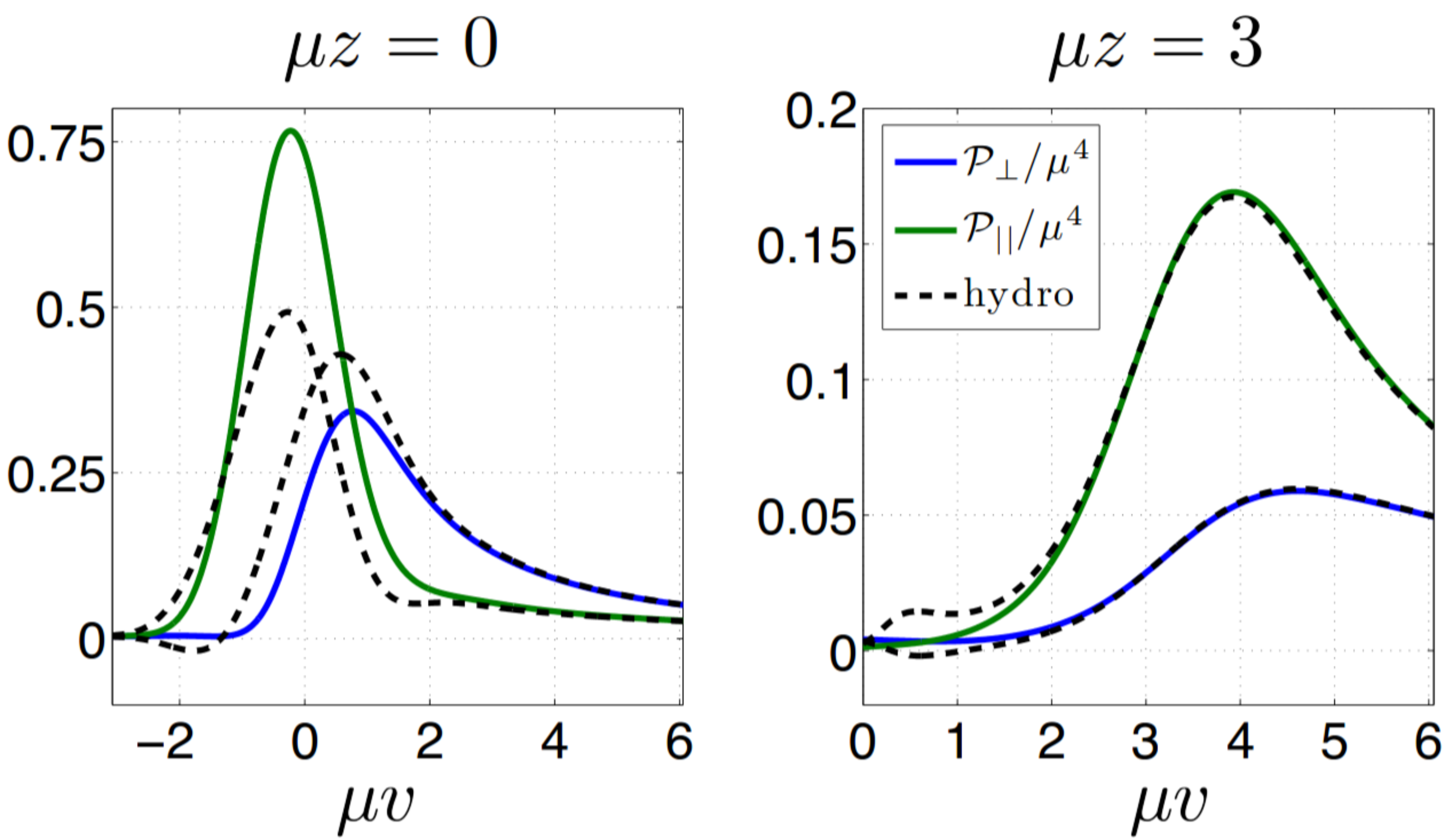
HIC is modeled by two colliding sheets of energy with infinite extend in transverse direction and Gaussian profile in beam direction. [Chesler, Yaffe, 1011.3562]

$$ds^2 = -A(r,v,y)dv^2 + 2dv(dr + F(r,v,y)dy) + \Sigma(r,v,y)^2(e^{-2B(r,v,y)}dy^2 + e^{B(r,v,y)}d\vec{x}^2)$$





Hydrodynamization of Shocks

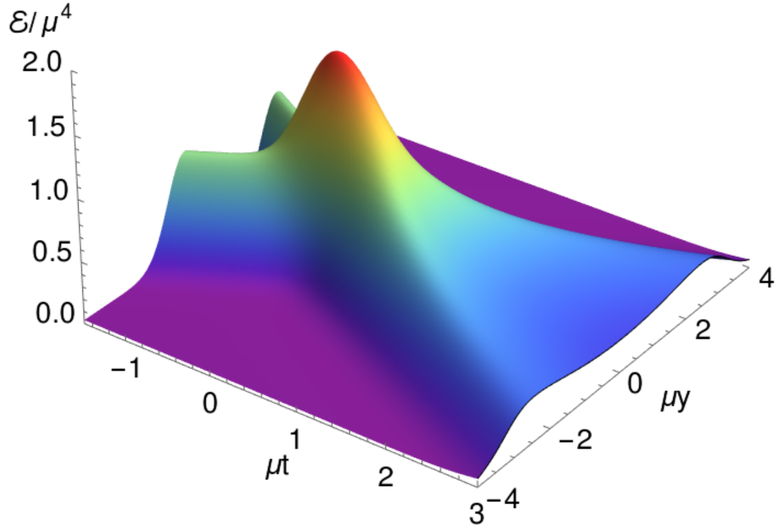


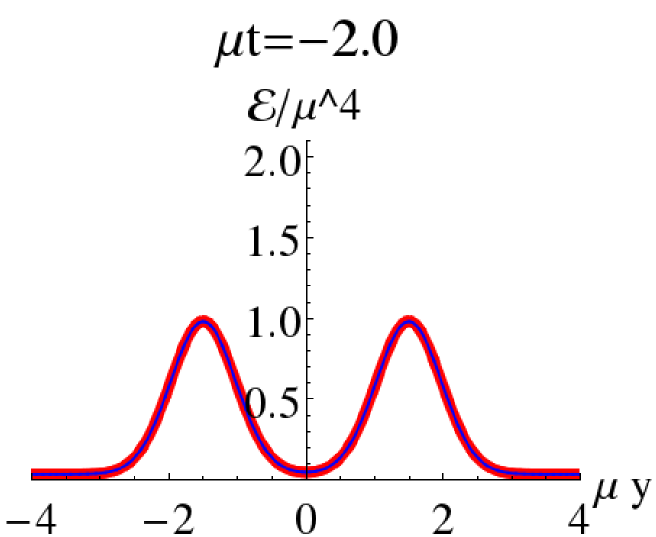
[Chesler, Yaffe, 1011.3562]

Wide vs. Narrow Shocks

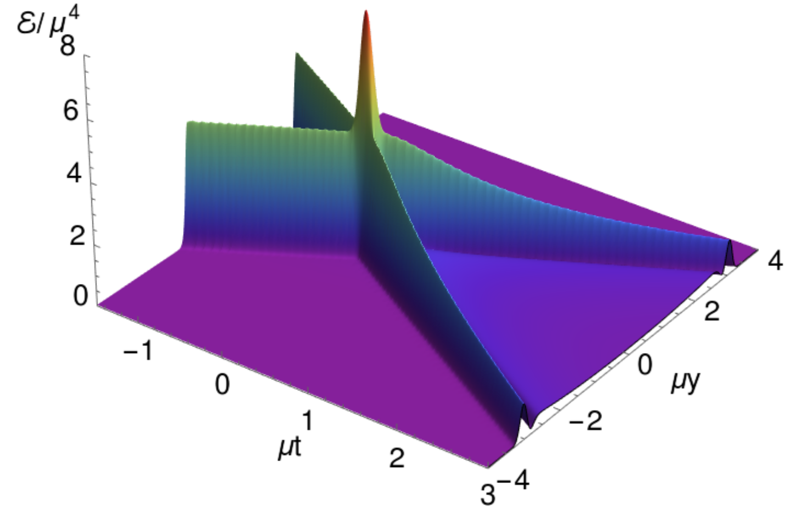
Two qualitatively different dynamical regimes [Solana, Heller, Mateos, van der Schee, 1305.4919]

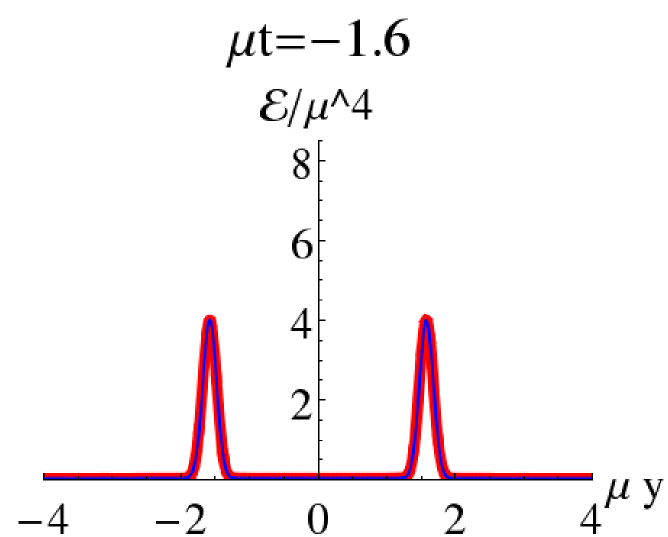
- Wide shocks: full stopping





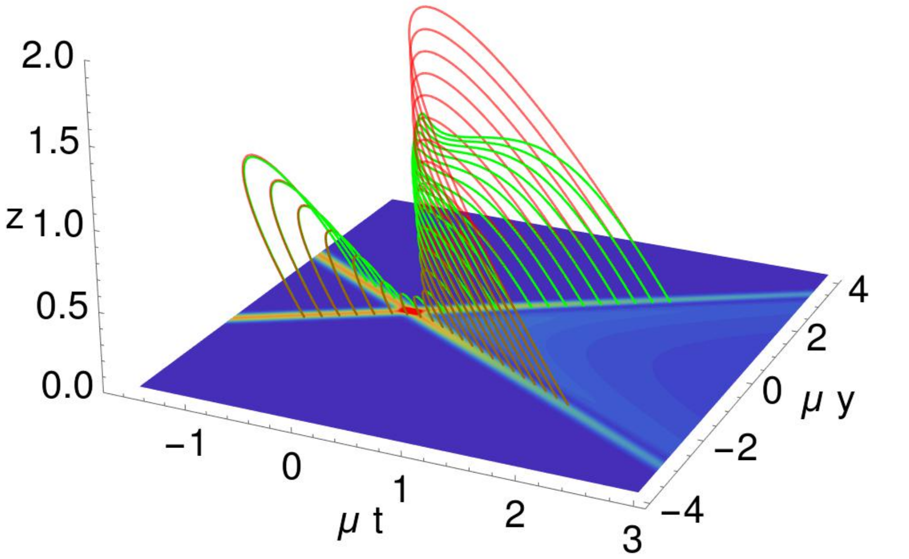
- Narrow shocks: transparency



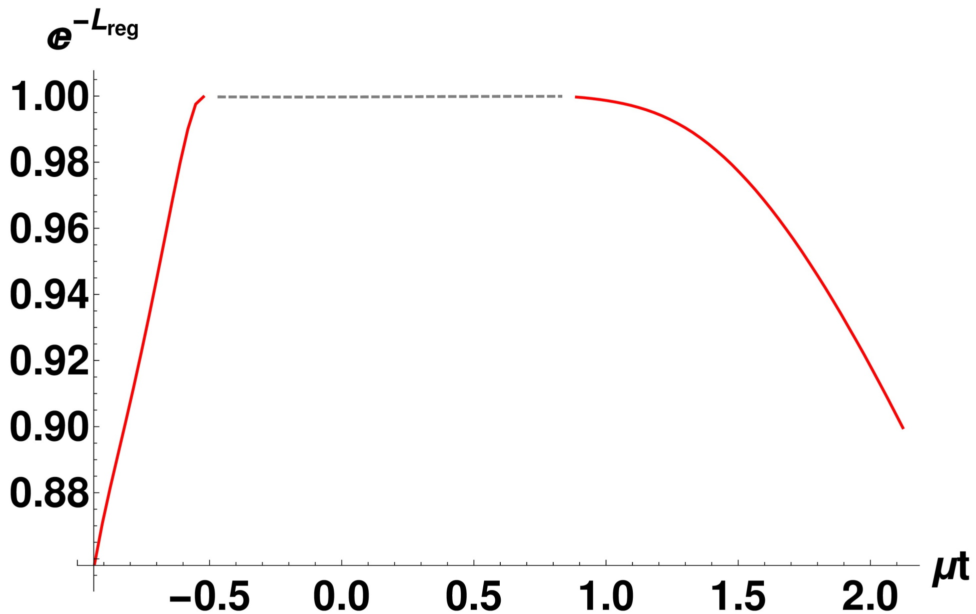


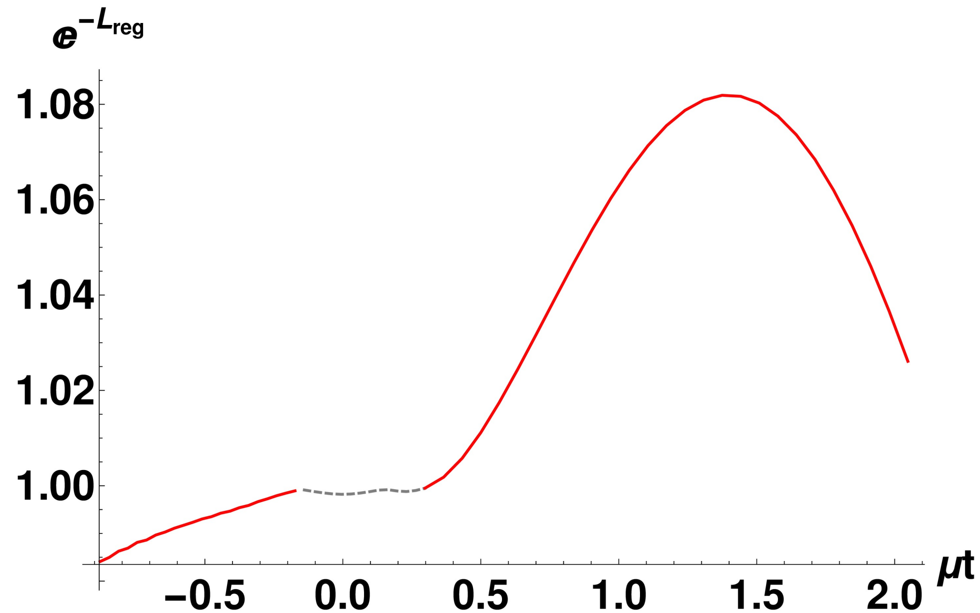
Correlations Between Shocks

$$\langle \mathcal{O}(t,x)\mathcal{O}(t,x')\rangle \approx e^{-\Delta L}$$

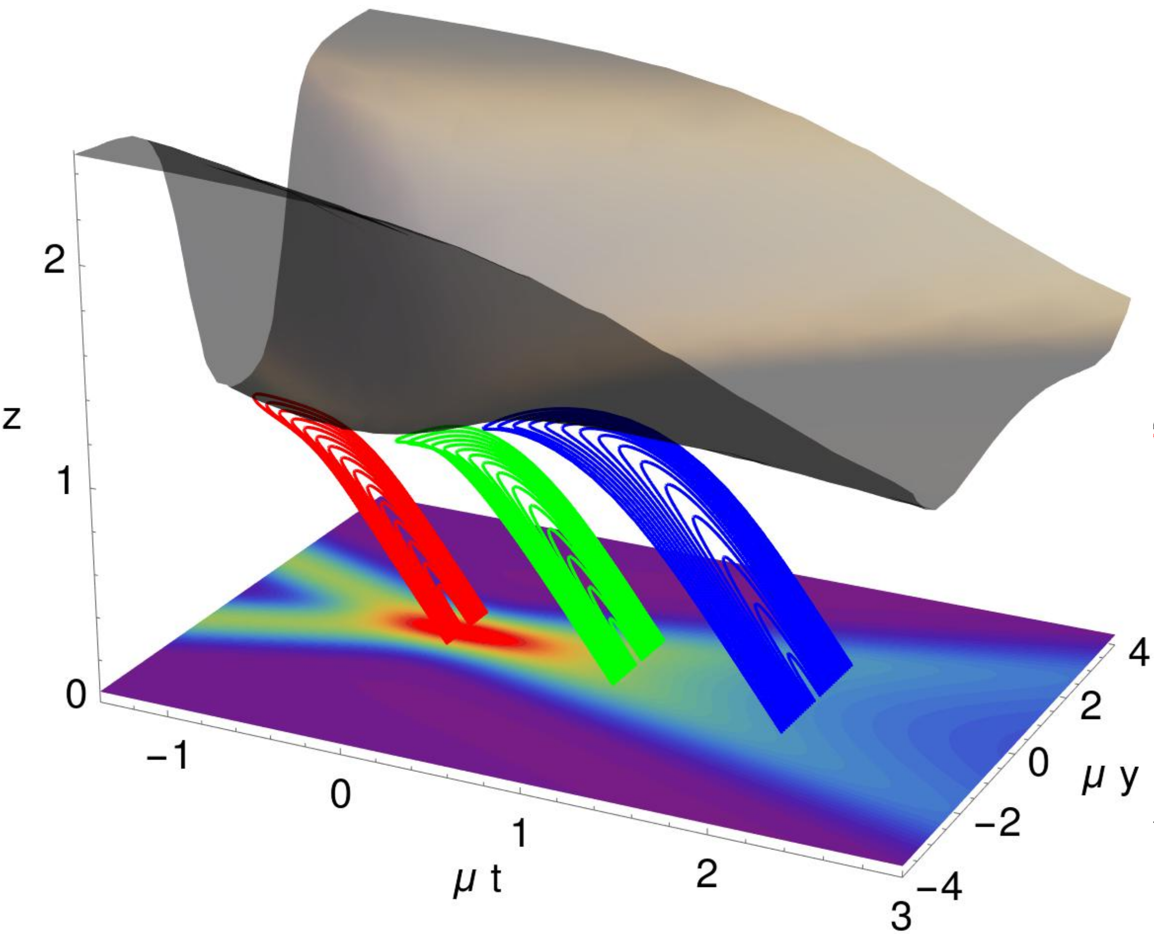


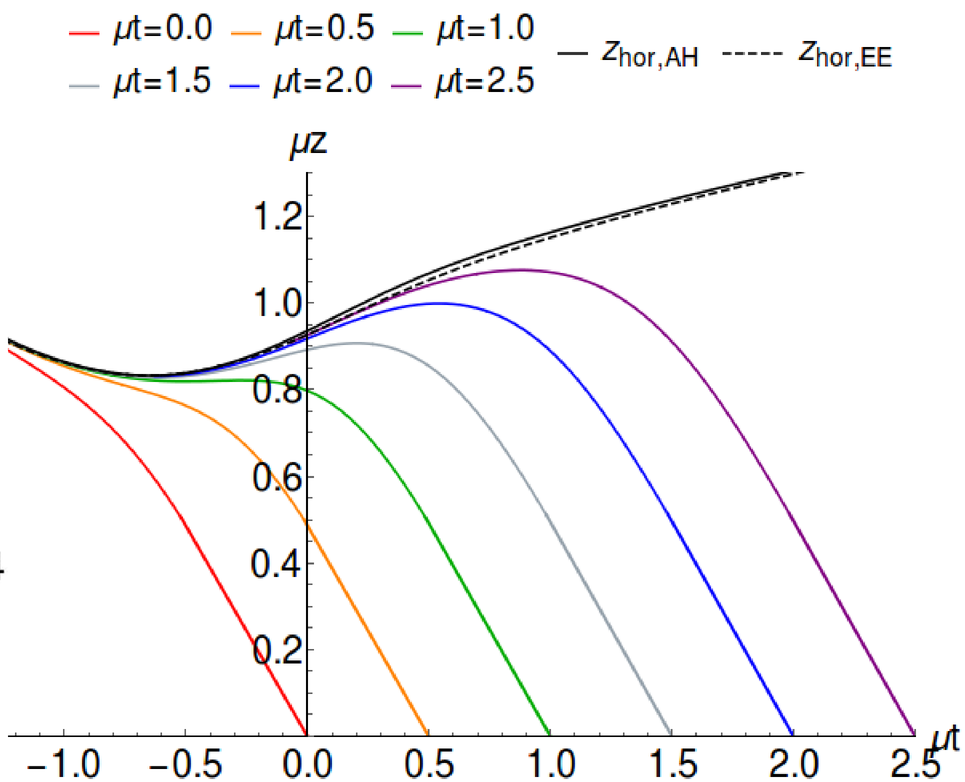
[CE, Grumiller, van der Schee, Stanzer, 1609.03676]





Extremal Surfaces

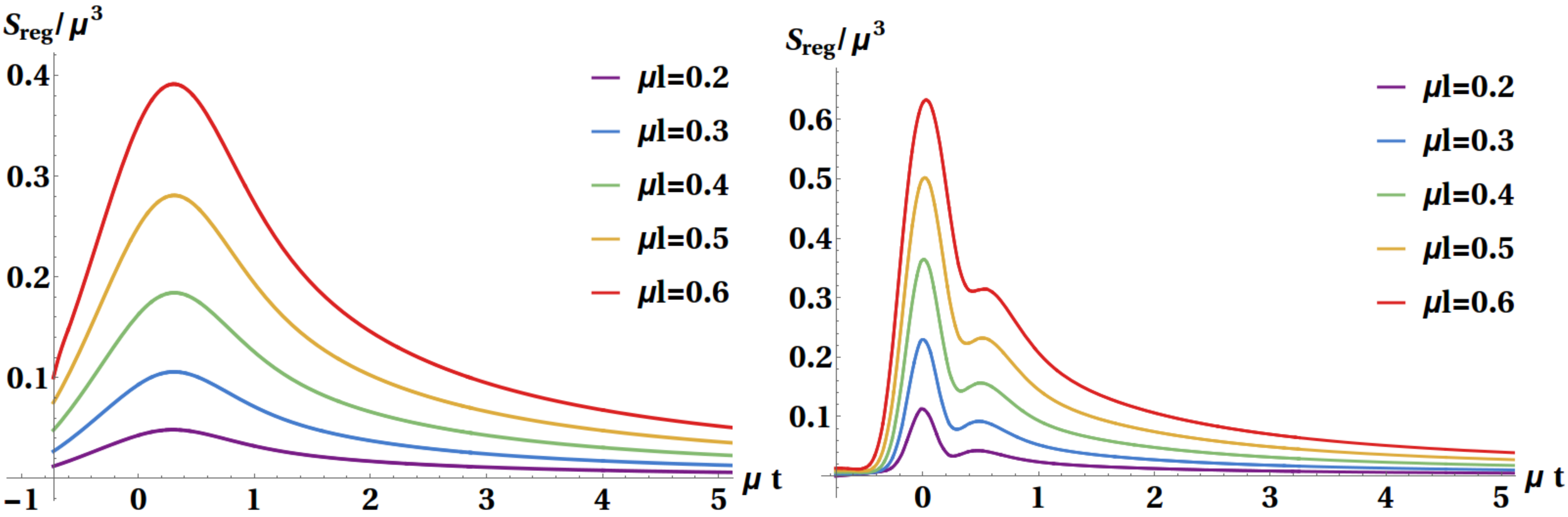




[CE, Grumiller, van der Schee, Stanzer, 1609.03676]

Time Evolution of Entanglement Entropy

$$S_A = \frac{\text{Area}(\Sigma)}{4G_N}$$



[CE, Grumiller, van der Schee, Stanzer, 1609.03676]

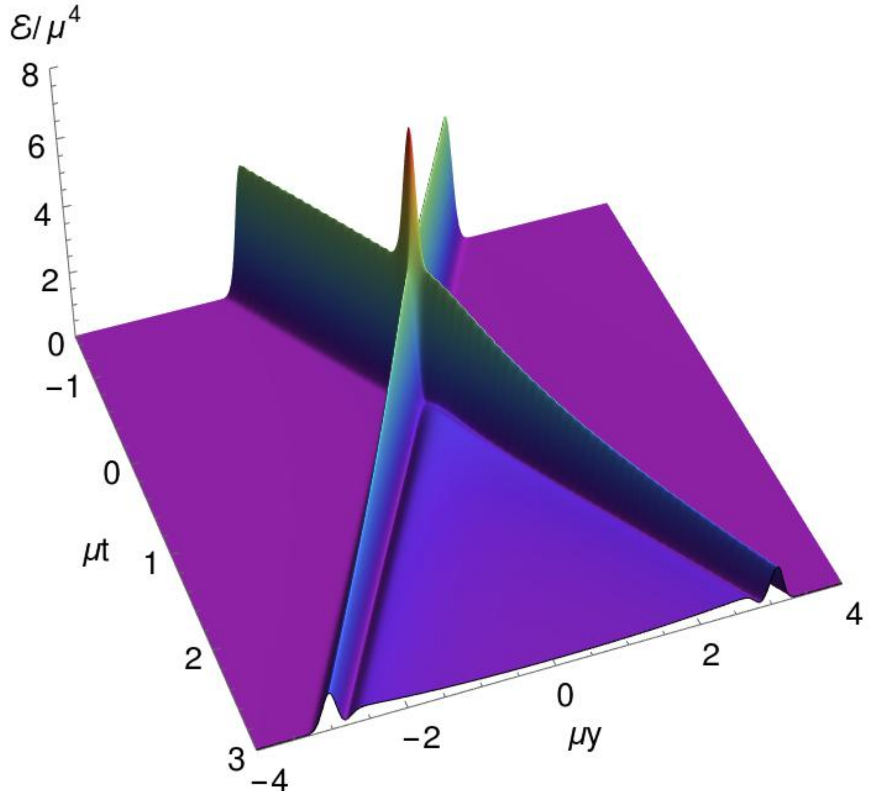
Null Energy Condition in Shock Wave Collisions

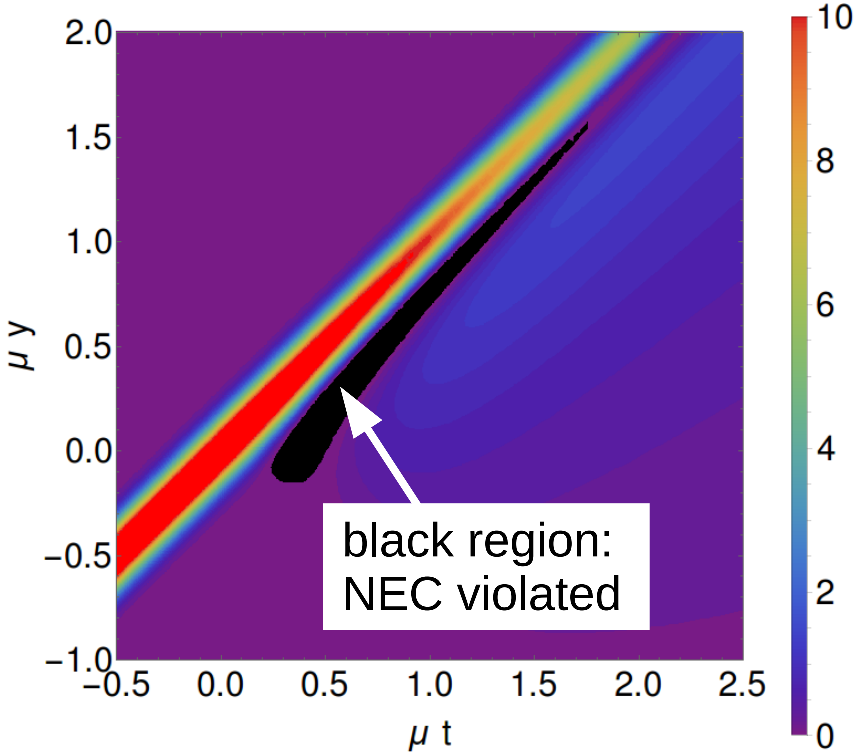
- Narrow shock wave collisions can violate the null energy condition (NEC)

[Arnold, Romatschke, van der Schee, 1408.2518]

$$T_{\mu\nu}k^\mu k^\nu \geq 0\,, \quad k_\mu k^\mu = 0\,.$$

black region: $T_{\mu\nu}k^\mu k^\nu < 0$





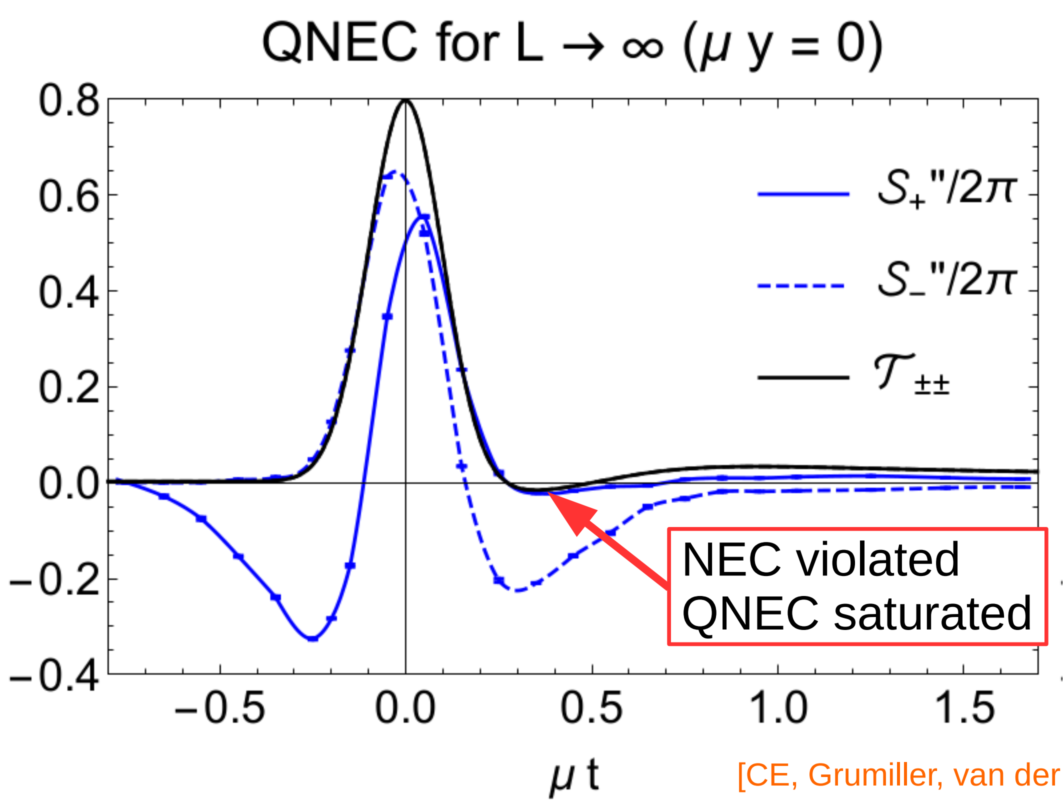
[CE, Grumiller, van der Schee, Stanzer 1710.09837]

QNEC in Shock Wave Collisions

- Quantum null energy condition (QNEC) replaces classical NEC

[Bousso, Fisher, Koeller, Leichenauer, Wall 1509.02542]

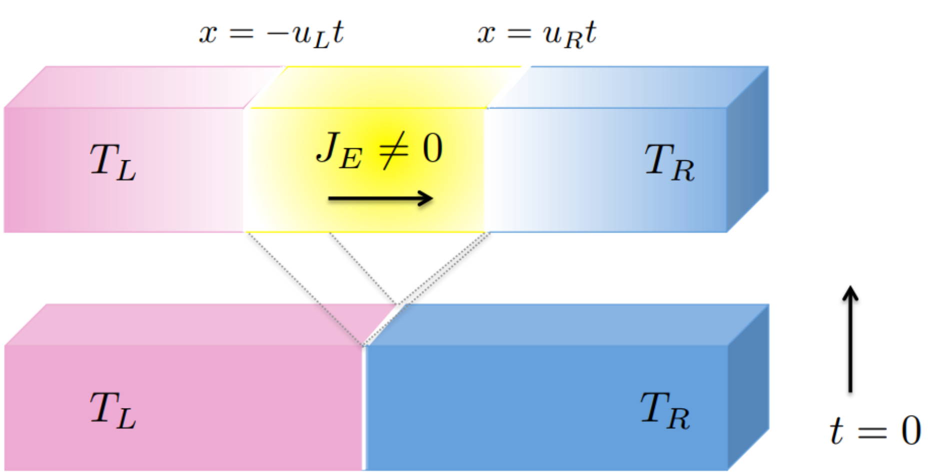
$$\langle T_{ab} k^a k^b \rangle \geq \frac{\hbar}{2\pi\sqrt{h}} S'' \qquad \forall k^2 = 0$$

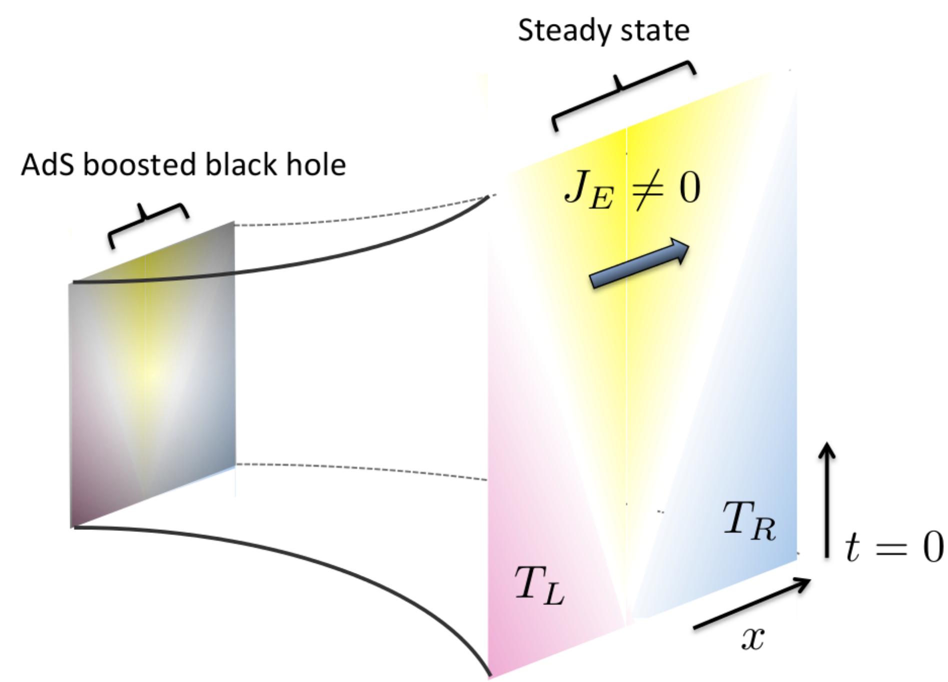


[CE, Grumiller, van der Schee, Stanzer 1710.09837]

Steady State Formation

- Thermal contact between strongly coupled quantum critical systems gives rise to a homogeneous steady state with non-vanishing energy flow.
- D=2: Steady state is described by Lorentz boosted equilibrium state.

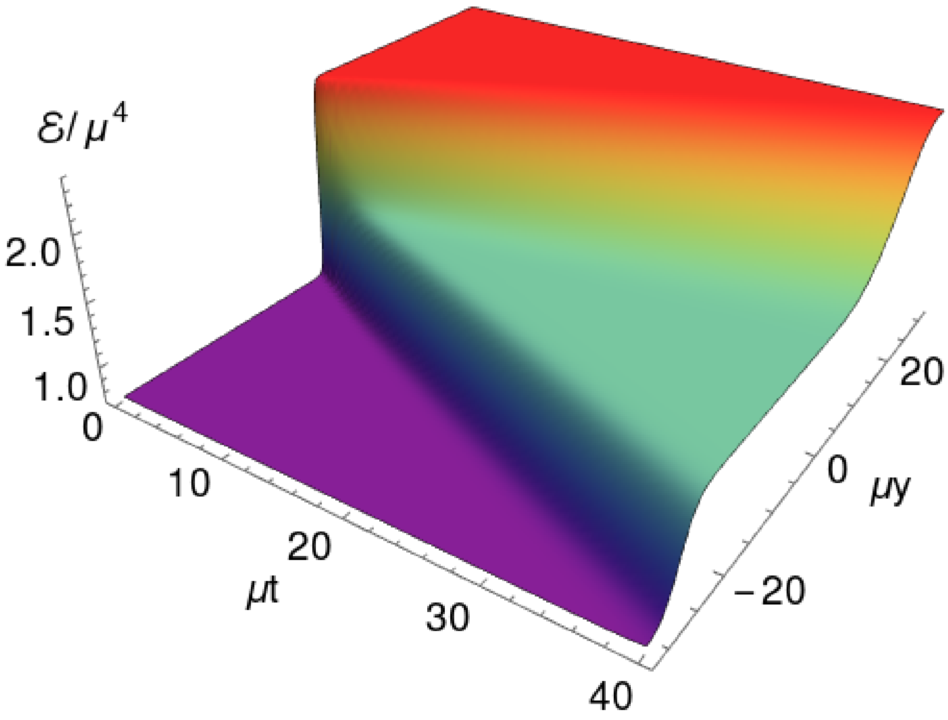


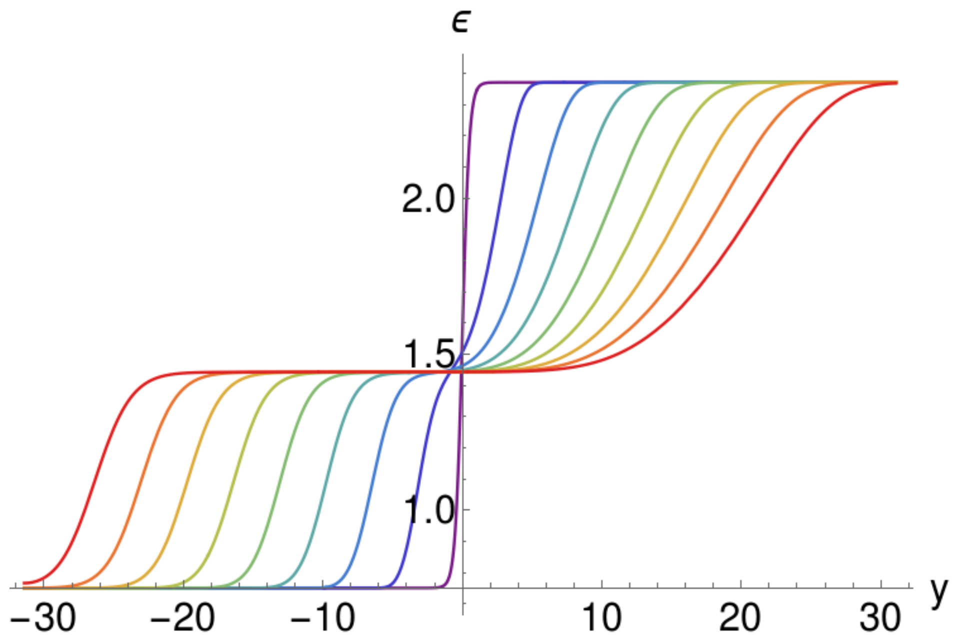


[Bhaseen, Doyon, Lucas, Schalm, 1311.3655]

Result for AdS5

- D>2: Shockwave moving to cold, rarefaction wave moving to warm.





[CE, Erdmenger, van der Schee, in progress]

Summary

- AdS/CFT is currently the only available tool to compute real time dynamics of strongly coupled gauge theories.
- Characteristic formulation in combination with spectral methods is an efficient numerical approach to numerical AdS/CFT.
- Simplest case: Homogeneous Isotropization, thermalizes, late time dynamics described by QNMs.
- Boost invariant case hydrodynamizes to universal hydro solution.
- Wide Shock waves show viscous hydrodynamic behavior even when pressure anisotropies are large.
- Wide and narrow shocks behave qualitatively different.
- Narrow shock waves can violate NEC but QNEC holds.