Lectures on Numerical Holography

Christian Ecker

Universiteit Utrecht

Universiteit Leiden

DKPI Summer School
Zwettl, 16–20 Sep. 2019

Plan of the Lectures

Day 1

- Lecture 1: Introduction to AdS/CFT
- Tutorial 1: Putting Einstein Equations on the Computer

Day 2

- Lecture 2: Holographic Heavy Ion Collisions
- Tutorial 2: Time Evolution of a $\mathcal{N} = 4$ SYM Plasma

Day 3

- Lecture+Tutorial 3: Entanglement Entropy

Useful References

Introduction to AdS/CFT and Applications:

- Introductory book on AdS/CFT: Ammon, Erdmenger, Cambridge University Press (2015)
- Condensed version in TASI lectures 2017 [1807.09872] by Erdmenger.
- Book on AdS/CFT applied to Heavy Ion Collisions: Casalderray-Solana, Liu, Mateos, Rajagopal, Wiedemann, Cambridge University Press (2014), preprint: [1101.0618].
- Book on AdS/CMT: Zaanen, Sun, Liu, Schalm, Cambridge University Press (2016)
- Book on holographic entanglement entropy: Rangamani, Takayanagi, Lecture notes in Physics, Springer (2017), perprint: [1609.01287]

Numerics:

- Review on Characteristic Method for Numerical Holography: Chesler, Yaffe [1309.1439]
- Book on Spectral Methods: Boyd, Dover Publications (2001)

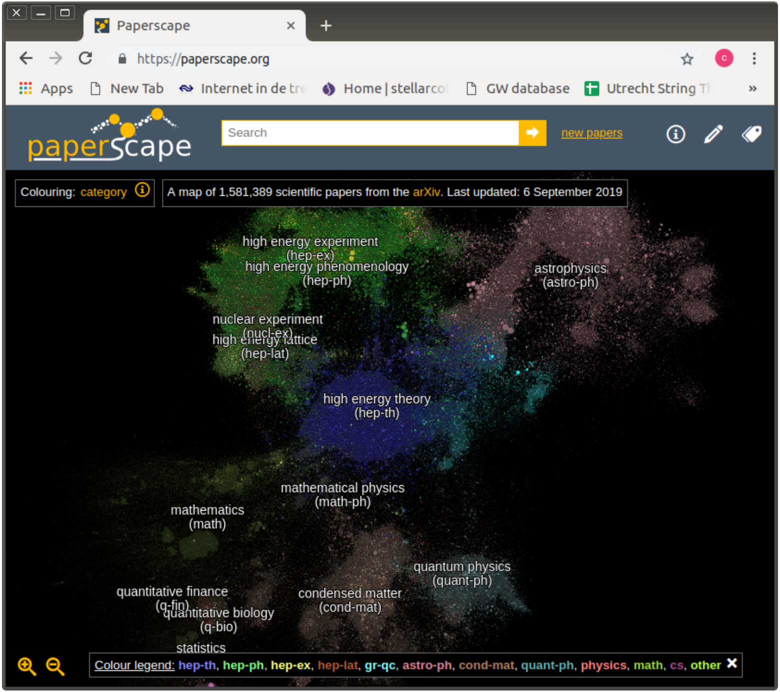
Code:

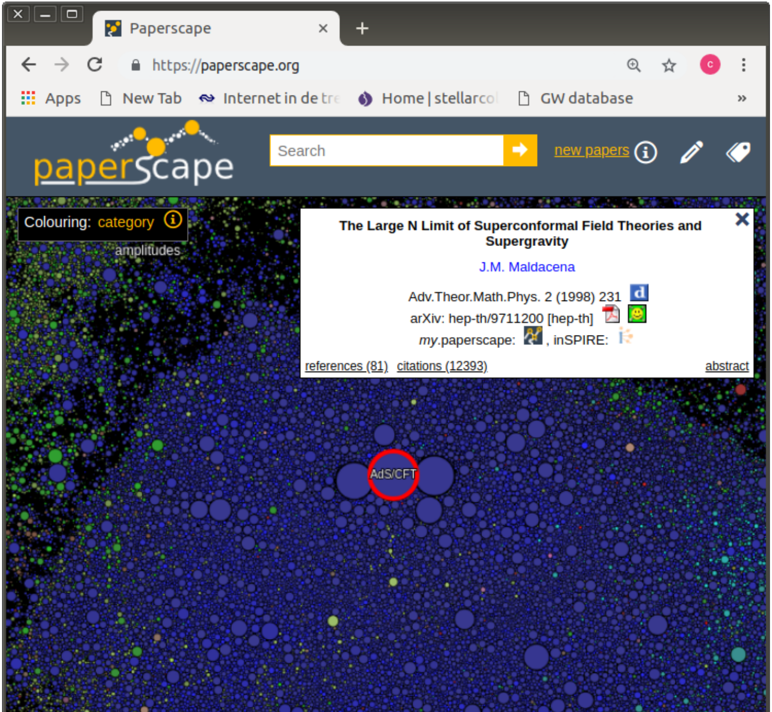
- Book by Trefethen has MATLAB code examples
- Homepage by van der Schee has instructive Mathematica codes.
- These slides and Mathematica files used in the tutorials on http://christianecker.com/.

Outline of Lecture 1

1. The AdS/CFT Correspondence
2. Computing Observables in AdS/CFT
3. Numerical Relativity on AdS

1. The AdS/CFT Correspondence


Paperscape
https://paperscape.org
Apps
New Tab
GW database
paperscape
Search
new papers
Colouring: category
A map of 1,581,389 scientific papers from the arXiv. Last updated: 6 September 2019
high energy experiment (hep-ex)
high energy phenomenology (hep-ph)
astrophysics (astro-ph)
nuclear experiment (nucl-ex)
high energy lattice (hep-lat)
high energy theory (hep-th)
mathematical physics (math-ph)
mathematics (math)
quantum physics (quant-ph)
quantitative finance (q-fin)
quantitative biology (q-bio)
condensed matter (cond-mat)
statistics
Colour legend: hep-th, hep-ph, hep-ex, hep-lat, gr-qc, astro-ph, cond-mat, quant-ph, physics, math, cs, other


Paperscape
https://paperscape.org
Apps
New Tab
GW database
paperscape
Search
new papers
Colouring: category
amplitudes
The Large N Limit of Superconformal Field Theories and Supergravity
J.M. Maldacena
Adv.Theor.Math.Phys. 2 (1998) 231
arXiv: hep-th/9711200 [hep-th]
my.paperscape: , inSPIRE:
references (81) citations (12393)
abstract
AdS/CFT

https://paperscape.org
Apps
New Tab
GW database
paperscape
?cites hep-th/9711200
Showing 12389 results. Clear search result
new papers
Colouring: category
high energy experiment
(hep-ex)
high energy phenomenology
(hep-ph)
astrophysic
(astro-ph)
nuclear experiment
(nucl-ex)
high energy lattice
(hep-lat)
high energy theory
(hep-th)
mathematical physics
(math-ph)
mathematics
(math)

What makes AdS/CFT special?

- One of the major theoretical developments in the last twenty years.
- It relates conformal field theories (CFTs) to higher dimensional gravity in Anti de Sitter (AdS) space.
- It is a strong-weak duality: if field theory is strongly coupled the gravity theory is weakly coupled and vice versa.
- Provides a framework for calculating observables in strongly coupled gauge theories which are generically hard to solve.
- This is for instance relevant for QGP in heavy ion collisions, strongly correlated condensed matter systems, neutron stars, ...
- Provides unexpected links between previously unrelated areas of physics, e.g.: General Relativity and Quantum Information

AdS/CFT Correspondence

Type IIB string theory on $AdS_5 \times S_5$

$=$

SU(N) $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory on $\mathcal{M}_4$

[Maldacena [hep-th/9711200]]

- The correspondence relates the parameters of the two theories

$$g_{YM}^2 = 2\pi g_s \,, \quad \lambda = 2 g_{YM}^2 N = L^4/\ell_s^4 \,.$$

- The correspondence is conjectured to hold for any value of the 't Hooft coupling $\lambda = 2g_{YM}^2 N$ and rank of gauge group N.
- **Supergravity limit:** Assuming point like strings ($\ell_s \to 0$) and small coupling ($g_s \ll 1$) reduces the string theory side to classical supergravity.
- This corresponds to the $N \to \infty$ and $\lambda \to \infty$ limit on the field theory side

$$\ell_s^4 = \frac{L^4}{\lambda} \xrightarrow{\lambda \text{ large}} 0 \,, \qquad g_s = \frac{\lambda}{2N} \xrightarrow{N \to \infty} 0 \,.$$

$\Rightarrow$ Observables in a strongly coupled field theory (hard) can be obtained from a classical gravity calculation (easy).

The Field Theory Side: $\mathcal{N} = 4$ SYM

- $\mathcal{N} = 4$ SYM is a SUSY field theory with a gauge field, scalars and fermions in the adjoint representation of the gauge group.
- The β function vanishes exactly to all orders in perturbation theory $\Rightarrow$ it is superconformal[1], hence no running coupling, no confinement and no chiral symmetry breaking.
- This is quite different from QCD which is has no SUSY, has fermions (quarks) in the fundamental representation, non-trivial β function, confinement and a chirally broken vacuum.
- However, at $T > T_c \approx 170$MeV some of the qualitative differences become unimportant and many authors used it as toymodel for the quark-gluon plasma produced in heavy ion collisions.
- Finite T explicitly breaks SUSY.
- Above T_c QCD is deconfined and the chiral condensate melts away.

[1] $\mathcal{N} = 4$ is the maximum number of SUSY generators in D=4 for spin $\leqslant 1$, i.e. without gravity, which makes the theory maximally supersymmetric.

The Gravity Side: AdS Spacetime

- AdS_{d+1} is a maximally symmetric solution of the Einstein equations with negative cosmological constant Λ and negative curvature radius L.

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0\,, \quad \Lambda = -\frac{d(d-1)}{2L^2}$$

- This is different form our universe which is well described by de Sitter space with small positive cosmological constant $\Lambda = 1.1056 \times 10^{-52}\,\mathrm{m}^{-2}$.
- In Poincaré patch[2] coordinates the line element of AdS reads

$$ds^2 = \frac{L^2}{r^2} dr^2 + \frac{r^2}{L^2} \underbrace{\left(-dt^2 + d\vec{x}^2\right)}_{\text{boundary metric}}$$

- AdS space has a timelike boundary at $r = \infty$, which for the Poincaré patch is Minkowski space. In the AdS/CFT context this is where the CFT lives.
- Asymptotic AdS spacetimes, like the AdS black brane, look only at $r \to \infty$ like AdS, but differ in the interior, e.g. by the presence of a BH horizon.

[2] The Poincaré patch covers only part of global AdS which has a compact boundary.

The Holographic Dictonary

- Every field on the gravity side corresponds is dual to a gauge invariant operator on the field theory side.

Gravity Side	Gauge Theory Side
metric $g_{\mu\nu}$	$T^{\mu\nu}$ stress tensor
scalar field ϕ	$\mathcal{O}$ scalar operator
gauge field A_μ	J^μ global sym. current
fermion field $\boldsymbol{\psi}$	$\mathcal{O}_\psi$ fermionic operator
...	...

- Operator $\mathcal{O}$ of dimension Δ sourced by $J = \phi_0$ dual to scalar with mass m

$$\phi(r,t,\vec{x}) \sim J(t,\vec{x})r^{-d+\Delta} + \langle\mathcal{O}(t,\vec{x})\rangle r^{-\Delta}, \quad \Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2L^2}, \quad m^2L^2 \geqslant -\frac{d^2}{4}$$

- Geometry in the bulk corresponds to a state in the field theory: e.g.: black hole geometries correspond to finite temperature states with T equal the Hawking temperature of the BH.

- **Holographic thermalization:** The dynamical process of BH formation gets mapped to the relaxation and thermalization of excited states in the field theory.

Computing Observables

- Expectation values of correlators of gauge invariant operators in the CFT follow from variations of the renormalized gravity action $S_{ren}[\phi]$ w.r.t. the boundary values ϕ_0 of the dual fields ϕ

$$\langle \mathcal{O}(x_1)\dots\mathcal{O}(x_n)\rangle_{CFT} = \frac{\delta^n S_{ren}[\phi]}{\delta\phi_0(x_1)\dots\delta\phi_0(x_n)}\bigg|_{\phi_0=0}.$$

- For instance the holographic stress tensor follows from

$$\langle T_{\mu\nu}(x)\rangle = -\frac{2}{\sqrt{g_{(0)}}}\frac{\delta S_{ren}}{\delta g_{(0)}^{\mu\nu}(x)}.$$

- Let's do this for a simple gravity action without bulk matter

$$S = -\frac{1}{16\pi G_N}\int d^{d+1}x\sqrt{g}\left(R + \frac{d(d-1)}{L^2}\right) - \frac{1}{8\pi G_N}\int d^d x\sqrt{\gamma}K.$$

Fefferman-Graham Expansion

- The bulk metric G_{MN} can be expanded in the radial coordinate ρ

$$ds^2 = G_{MN} dx^M dx^N = L^2 \left(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{\mu\nu}(\rho, x) dx^\mu dx^\nu \right)$$

- **Fefferman-Graham theorem:** If G_{MN} satisfies the Einstein equations, then $g_{\mu\nu}$ has the following expansion

$$g_{\mu\nu}(\rho, x) = g_{(0)\mu\nu}(x) + \rho g_{(2)\mu\nu}(x) + \ldots + \rho^{d/2} \left(\log(\rho) h_{(d)\mu\nu}(x) + g_{(d)\mu\nu}(x) \right) + \ldots$$

- The expansion coefficients are obtained by solving the Einstein equations order by in ρ, where the leading ones only depend on the boundary metric

$$g_{(2)\mu\nu}(x) = \frac{L}{d-2} \left(R_{(0)\mu\nu} - \frac{1}{2(d-1)} R_{(0)} g_{(0)\mu\nu} \right)$$

- **Important remarks:**
  Logarithms are related to conformal anomalies and only present for even d.
  Coefficients of order $\geqslant d$ can only be extracted from full bulk solution.

Holographic Renormalization

- Putting the asymptotic expansion into the action and evaluating it at cutoff $\epsilon \ll 1$ gives

$$S_\epsilon = -\frac{1}{16\pi G_N}\int d^d x\sqrt{g_{(0)}}\left(a_{(0)}\epsilon^{-d/2} + a_{(2)}\epsilon^{-d/2+1} + \ldots - \log a_{(d)}\epsilon\right) + S_{finite}$$

- To renormalize the action we have to add appropriate counter terms[3]

$$S_{ren} = \lim_{\epsilon\to 0}(S_\epsilon + S_{ct})$$

- Varying the renormalized action w.r.t. the boundary metric gives the stress tensor

$$\langle T_{\mu\nu}(x)\rangle = -\frac{2}{\sqrt{g_{(0)}}}\frac{\delta S_{ren}}{\delta g^{\mu\nu}_{(0)}(x)}$$

- For $d = 4$ this gives the following expression for the holographic energy momentum tensor [de Haro, Skenderis, Solodukhin [hep-th/0002230]]

$$\langle T_{\mu\nu}\rangle = \frac{4}{16\pi G_N}\left(g_{(4)\mu\nu} + \frac{1}{8}\left(\mathrm{Tr}g^2_{(2)} - (\mathrm{Tr}g_{(2)})^2\right)g_{(0)\mu\nu} - \frac{1}{2}(g^2_{(2)})_{\mu\nu} + \frac{1}{4}g_{(2)\mu\nu}\mathrm{Tr}g_{(2)}\right)$$

[3] The counterterms are usually ambiguous, and a specific choices for S_{ct} correspond to specific renormalization schemes.

2. Numerical Relativity on AdS

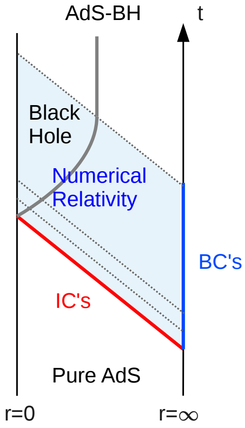
Characteristic Formulation

- AdS is not globally hyperbolic, i.e. has no Cauchy slice: Need ICs+BCs to obtain well defined initial value problem.
- Light-like slicing results in characteristic formulation of GR, realized by generalized Eddington-Finkelstein (EF) coordinates

$$ds^2 = 2dvdr + \frac{r^2}{L^2} g_{\mu\nu}(r, x^\mu) dx^\mu dx^\nu.$$

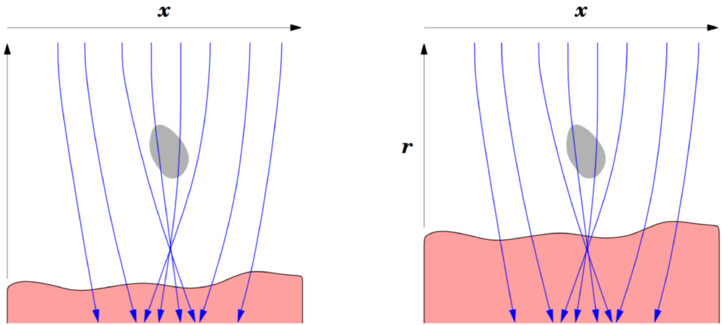
- EF coordinates are regular across black hole horizons.
- Residual gauge freedom can be used to fix radial position of the apparent horizon

$$r \to \bar{r} \equiv r + \xi(x^\mu).$$



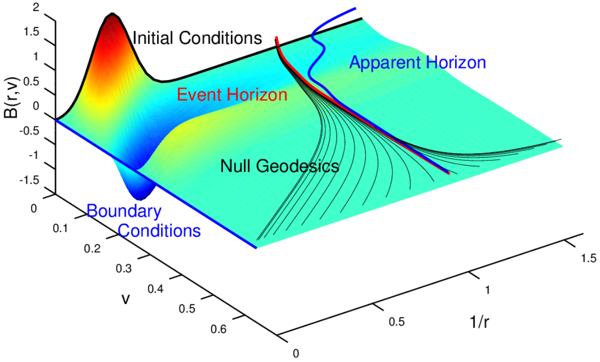
Caustics

- Penrose focusing theorem: *"Matter focuses light."*
- Coordinate lines are light-like geodesics which can form caustics. Caustics are coordinate singularities destroying the coordinate system.
- Increase regulator energy density to hide caustics behind horizon.



Anisotropic AdS_5 Black Brane

- Simplest non-trivial case: homgeneous, anisotropic AdS_5 black brane.
$$ds^2 = -A(r,v)dv^2 + 2dvdr + S^2(r,v)\left(e^{-2B(r,v)}dy^2 + e^{B(r,v)}d\vec{x}^2\right)$$
- BCs: Minkowski boundary $ds^2_{\text{bdry}}| = -dt^2 + d\vec{x}^2$
- ICs: Warp factor $B(r,v_0) = \frac{6.6}{r^4}e^{-\left(\frac{1}{r}-\frac{1}{4}\right)^2}$, Energy $a_4 = -1 \to T_{eq} = \frac{1}{\pi}$
- Models isotropization of an intially anisotropic $\mathcal{N} = 4$ SYM plasma.



Characteristic Bulk Equations

- For this simple example we need to solve the 5D Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \quad \text{s.t.} \quad ds^2|_{r\to\infty} = r^2(-dt^2 + d\vec{x}^2)\,.$$

- Derivatives along ingoing (prime) and outgoing (dot) null geodesics

$$h' \equiv \partial_r h\,, \quad \dot{h} \equiv \partial_v h + \tfrac{1}{2} A \partial_r h\,.$$

- Einstein Equations decouple into a nested set of ODEs

IC's: $B_{v=v_0} \longrightarrow$

$$S'' + \tfrac{1}{2} B'^2 S = 0 \qquad (1)$$

$$S(\dot{S})' + 2S'\dot{S} - 2S^2 = 0 \qquad (2)$$

$$S(\dot{B})' + \tfrac{3}{2}(S'\dot{B} + 2B'\dot{S}) = 0 \qquad (3)$$

$$A'' + 3B'\dot{B} - 12S'\dot{S}/S^2 + 4 = 0 \qquad (4)$$

$$\ddot{S} + \tfrac{1}{2}(\dot{B}^2 S - A'\dot{S}) = 0 \qquad (5)$$

$\xrightarrow{A}$ $\dot{B} = \partial_v B + \frac{1}{2} A \partial_r B$ $\xrightarrow{\partial_v B}$ $B_{(v+\Delta v)} = B_{(v)} + \Delta v \partial_v B_{(v)}$ $\longrightarrow$ (1)

- Use constraint (1) to prepare ICs, constraint (5) to monitor accuracy.

Field Redefinitions

- Use inverse radial coordinate $z = \frac{1}{r}$ where the AdS boundary is at $z = 0$.
- Finite fields are obtained by factoring out the analytically known divergent near boundary part

$$A(z,v) \to \frac{1}{z^2} + z\tilde{A}(z,v)\,, \quad S(z,v) \to \frac{1}{z} + z^2\tilde{S}(z,v)\,, \quad B(z,v) \to z^3\tilde{B}(z,v)\,.$$

- Redefinitions are tailored to read off the normalizable modes

$$b_4(v) = \tilde{B}'(0,v)\,, \quad a_4 = \tilde{A}'(0,v)$$

- On each slice we have to solve boundary value problems (BVP) with BCs fixed by the near boundary expansion, e.g. eq.(4) takes the form

$$\tilde{A}'' + \frac{4}{z}\tilde{A}' + \frac{2}{z^2}\tilde{A} = j_A\,, \quad \text{s.t.} \quad \tilde{A}(0,v_0) = 0\,, \tilde{A}'(0,v_0) = a_4$$

- In Lecture 2 we will learn how to solve such BVPs efficiently using spectral methods.

Time Stepping

- Simplest way: 1st order Euler method

$$y_i(t + \Delta t) = y_i(t) + \Delta t \cdot y_i'(t).$$

- More accurate: 4th order Adams-Bashforth

$$y_i(t+4\Delta t) = y_i(t+3\Delta t) + \frac{\Delta t}{24}\left(55y_i'(t+3\Delta t) - 59y_i'(t+2\Delta t) + 37y_i'(t+\Delta t) - 9y_i'(t)\right).$$

- Requires derivatives at three previous time steps → have to use lower order method for the first steps (lower accuracy).

Near Boundary Analysis

- Near $r \to \infty$ the Einstein equations have the power series solution

$$A(r,v) = r^2 + \frac{a_4(v)}{r^2} + \frac{a_4'(v)}{2r^3} + \mathcal{O}(r^{-4})\,,$$
$$S(r,v) = r - \frac{a_4'(v)}{20r^4} - \frac{b_4(v)^2}{7r^7} + \mathcal{O}(r^{-8})\,,$$
$$B(r,v) = \frac{b_4(v)}{r^4} + \frac{b_4'(v)}{r^5} + \mathcal{O}(r^{-6})\,,$$

- $a_0(v), s_0(v), b_0(v)$ are fixed by BCs: $ds^2|_{r\to\infty} = r^2(-dt^2 + d\vec{x}^2)$
- $b_4(v)$ needs to be extracted form the numerical solution.
- $a_4(v)$ and $b_4(v)$ contain information on the field theory stress tensor

$$\langle T^{\mu\nu}\rangle = \frac{N_c^2}{2\pi^2}\,\mathrm{diag}\left[\mathcal{E},\ P_\parallel(t),\ P_\perp(t),\ P_\perp(t)\right]$$

$$\mathcal{E} = -\frac{3}{4}a_4\,, \qquad P_\parallel(t) = -\frac{1}{4}a_4 - 2b_4(t)\,, \qquad P_\perp(t) = -\frac{1}{4}a_4 + b_4(t)$$

- Energy conservation at 5$^{\text{th}}$ order: $a_4'(v) = 0 \to a_4(v) = \text{const.}$

Summary of Lecture 1

- The AdS/CFT correspondence relates strongly coupled $\mathcal{N} = 4$ SYM theory to classical gravity on AdS.
- The basic method to compute observables, e.g. the stress tensor, in AdS/CFT is holographic renormalization. This requires to solve the Einstein equations perturbatively close to the boundary.
- The characteristic formulation reduces the Einstein equations (PDEs) to a nested set of ODEs. This is drastically simplifies the numerics.
- In the Tutorial 1 this afternoon we will use Mathematica to bring the Einstein equations into characteristic form, perform the near boundary analysis and derive the stress tensor.

Outline of Lecture 2

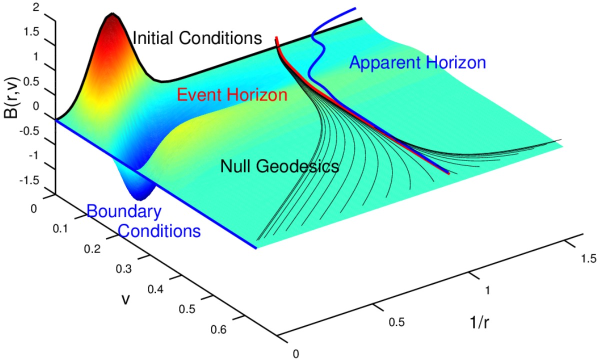
1. Spectral Methods
2. Homogeneous Isotropization
3. Boost Invariant Hydrodynamization
4. Shockwave Collisions

Recap: Anisotropic AdS_5 Black Brane

- Simplest non-trivial case: homgeneous, anisotropic AdS_5 black brane.

$$ds^2 = -A(r,v)dv^2 + 2dvdr + S^2(r,v)\left(e^{-2B(r,v)}dy^2 + e^{B(r,v)}d\vec{x}^2\right)$$

- BCs: Minkowski boundary $ds^2_{\mathrm{bdry}}| = -dt^2 + d\vec{x}^2$
- ICs: Warp factor $B(r,v_0) = \frac{6.6}{r^4} e^{-\left(\frac{1}{r}-\frac{1}{4}\right)^2}$, Energy $a_4 = -1 \rightarrow T_{eq} = \frac{1}{\pi}$
- Models isotropization of an intially anisotropic $\mathcal{N} = 4$ SYM plasma.



Recap: Anisotropic AdS$_5$ Black Brane

- We have to solve the 5D Einstein equations

IC's: $B_{v=v_0} \longrightarrow$

$$S'' + \tfrac{1}{2}B'^2 S = 0 \quad (1) \quad \longleftarrow \quad B_{(v+\Delta v)} = B_{(v)} + \Delta v \partial_v B_{(v)}$$

$$S(\dot{S})' + 2S'\dot{S} - 2S^2 = 0 \quad (2)$$

$$\uparrow \partial_v B$$

$$S(\dot{B})' + \tfrac{3}{2}(S'\dot{B} + 2B'\dot{S}) = 0 \quad (3)$$

$$A'' + 3B'\dot{B} - 12S'\dot{S}/S^2 + 4 = 0 \quad (4) \quad \xrightarrow{A} \quad \dot{B} = \partial_v B + \tfrac{1}{2}A\partial_r B$$

$$\ddot{S} + \tfrac{1}{2}(\dot{B}^2 S - A'\dot{S}) = 0 \quad (5)$$

- After using inverse radial coordinate $z = \frac{1}{r}$ and factoring out the analytically known divergent near boundary part, we are left over with a set of boundary value problems (BVP) on each slice, e.g. (4)

$$\tilde{A}'' + \frac{4}{z}\tilde{A}' + \frac{2}{z^2}\tilde{A} = j_A\,, \quad \text{s.t.} \quad \tilde{A}(0, v_0) = 0\,, \tilde{A}'(0, v_0) = a_4$$

- To evolve between slices we use 4$^{\text{th}}$ order Adams-Bashforth

$$y_i(t+4\Delta t) = y_i(t+3\Delta t) + \frac{\Delta t}{24}\left(55y_i'(t+3\Delta t) - 59y_i'(t+2\Delta t) + 37y_i'(t+\Delta t) - 9y_i'(t)\right).$$

Spectral Methods

- Expand in terms of Chebyshev polynomials

$$y(x) \approx \sum_{i=0}^{N-1} c_i T_i(x)\,, \quad T_i(\cos(x)) = \cos(ix)$$

- Highly efficient because of spectral convergence: error$\sim (\Delta x)^N$
- Spectral (Chebyshev) grid

$$x_i = \cos(i\pi/N)\,, \quad i = 0, \ldots, N$$

- Differentiation and integration on the spectral grid by matrix multiplication

$$y_i \equiv y(x_i)\,, \quad y_i' = D_{ij} y_j\,, \quad y_i'' = D_{ij}^2 y_j\,, \quad \int dx y_i = D_{ij}^{-1} y_j$$

- Spectral matrix, see book by Boyd 2001

$$D_{ij} = \frac{c_i}{c_j} \frac{(-1)^{i+j}}{x_i - x_j}\,, \quad c_0 = c_N = 2, c_i = 1\,, \quad i \neq j\,, \quad i,j = 0, \ldots, N$$

$$D_{00} = \frac{2N^2+1}{6}\,, \quad D_{NN} = -\frac{2N^2+1}{6}\,, \quad D_{jj} = -\frac{x_j}{2(1-x_j^2)}\,, \quad j = 1, \ldots, N-1$$

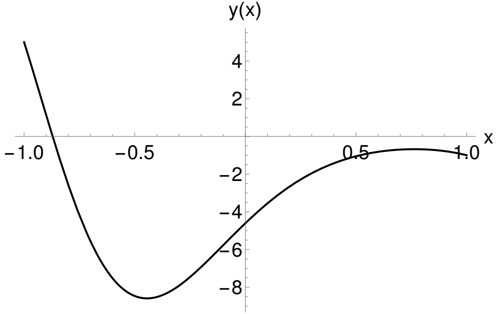
Boundary Value Problem (I)

- Consider the simple boundary value problem

$$y''(x) + y'(x) - 20xy(x) = 0\,, \qquad \mathrm{BC:}\quad y(-1) = 5\,, y(1) = -1$$

- It has a (not so simple) analytic solution

$$y(x) = \frac{e^{\frac{1}{2}(-x-1)}\left(\left(e\mathrm{Ai}\left(-\frac{79}{8\sqrt[3]{2}5^{2/3}}\right) + 5\mathrm{Ai}\left(\frac{81}{8\sqrt[3]{2}5^{2/3}}\right)\right)\mathrm{Bi}\left(\frac{80x+1}{8\sqrt[3]{2}5^{2/3}}\right) - \left(e\mathrm{Bi}\left(-\frac{79}{8\sqrt[3]{2}5^{2/3}}\right) + 5\mathrm{Bi}\left(\frac{81}{8\sqrt[3]{2}5^{2/3}}\right)\right)\mathrm{Ai}\left(\frac{80x+1}{8\sqrt[3]{2}5^{2/3}}\right)\right)}{\mathrm{Ai}\left(\frac{81}{8\sqrt[3]{2}5^{2/3}}\right)\mathrm{Bi}\left(-\frac{79}{8\sqrt[3]{2}5^{2/3}}\right) - \mathrm{Ai}\left(-\frac{79}{8\sqrt[3]{2}5^{2/3}}\right)\mathrm{Bi}\left(\frac{81}{8\sqrt[3]{2}5^{2/3}}\right)}$$



Boundary Value Problem (II)

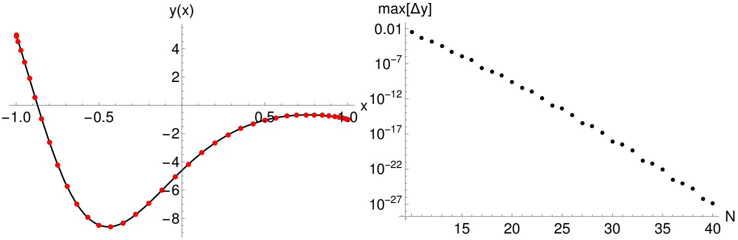
- Express the differential operator matrix and the source vector on the grid

$$L = \partial_x^2 + \partial_x - 20x \to L_{ij} = D_{ij}^2 + D_{ij} - 20\mathrm{diag}(x)_{ij}\,, \quad S = 0 \to S_i = (0, 0, \ldots, 0)$$

- Implement BCs: $y(-1) = y_1 = 5\,, y(1) = y_N = -1$

$$\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ L_{21} & L_{22} & L_{23} & \ldots & L_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ L_{N-1,1} & L_{N-1,2} & L_{N-1,3} & \ldots & L_{N-1,N} \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-1} \\ y_N \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}$$

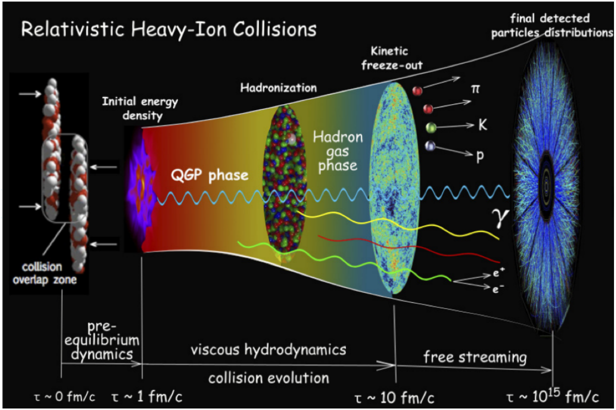
- The solution vector y_i is obtained by simple matrix inversion: $y_i = L_{ij}^{-1} S_j$



2. Holography and Heavy Ion Collisions

Heavy Ion Collisions

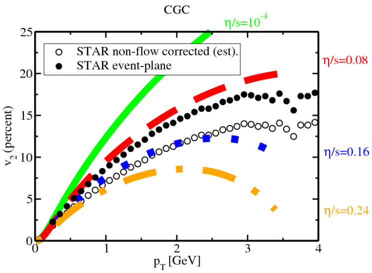
- Relativistic collisison of heavy ions (Au, Pb) at RHIC and LHC produce a deconfined state of quarks and gluons called quark-gluon plasma.



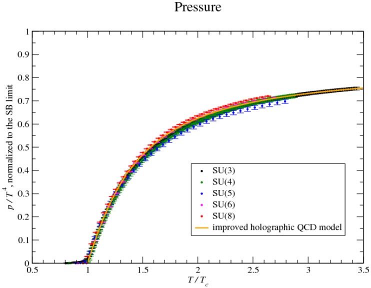
Holography for HICs

- The plasma is strongly coupled in the early stages of the collisions, which makes first principle calcualtions in perturbative QCD intractable.
- Experiments are only well described by viscouse hydrodynamics when assuming small η/s and early onset ($\approx 1 fm/c$) of hydro.
- Holography predicts such small viscosity $\eta/s = \frac{1}{4\pi}$ for $\mathcal{N} = 4$ SYM[4].

  [Policastro, Son, Starinets [hep-th/0104066]]
- Holography naturally realizes fast thermalization/hydrodynamization.
- Lattice calculations in thermal equilibrium suggest small influence of N.



[Luzum, Romatschke [0804.4015]]



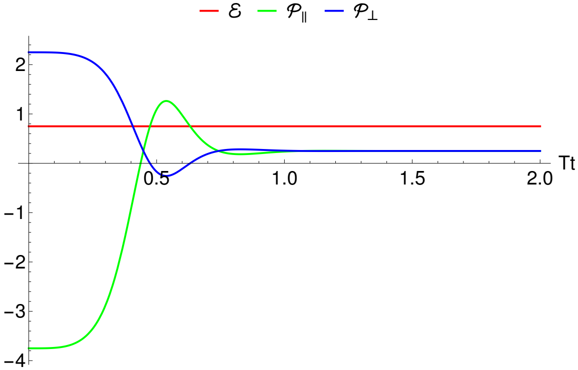
[Panero [0907.3719]]

[4] This result is universal for all theories with Einstein gravity dual.

Homogeneous Isotropization of $\mathcal{N} = 4$ SYM Plasma

- The QGP in HICs is initially highly anisotropic ($\mathcal{P}_\perp \neq \mathcal{P}_\parallel$).
- The anisotropic AdS_5 black brane introduced in Lecture 1 models the isotropization of a strongly coupled non-Abelian plasma.

$$\langle T^{\mu\nu} \rangle = \frac{N_c^2}{2\pi^2} \,\mathrm{diag}\left[\mathcal{E},\ \mathcal{P}_\parallel(t),\ \mathcal{P}_\perp(t),\ \mathcal{P}_\perp(t)\right]$$



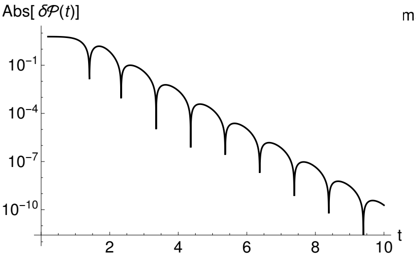
Quasinormal Ring Down

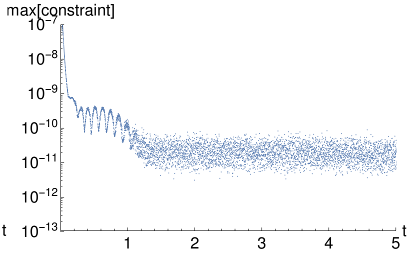
- The late time dynamics of the plasma is described by quasinormal modes of the black brane.

[Starinets hep-th/0207133]

$$\Delta\mathcal{P} \propto \mathrm{Re}\sum_n c_n e^{-\lambda_n t}\,, \quad \frac{\lambda_1}{\pi T} = 2.7467 + 3.1195 i\,, \quad \frac{\lambda_2}{\pi T} = 4.7636 + 5.1695 i\,, \ldots$$

- Accuracy with only 30 grid points is good enough to extract lowest QNM.





Boost Invariant Hydrodynamization

- The previous model was to simple to study the approach to hydro.
- The easiest way to realize a non-trivial flow is to use proper time and rapidity coordinates and assume boost invariance (=y-independence)

$$t = \tau \cosh y\,, \quad x_{\|} = \tau \sinh y\,, \quad ds^2_{\text{bdry}} = -d\tau^2 + \tau^2 dy^2 + d\vec{x}_{\perp}^2$$

$$ds^2 = -A(r,\tau)d\tau^2 + 2d\tau dr + S(r,\tau)^2(e^{-2B(r,\tau)}dy^2 + e^{B(r,\tau)}d\vec{x}_{\perp}^2)\,.$$

- Energy momentum tensor is diagonal

$$T_{\mu\nu} = \frac{N_c^2}{2\pi^2}\mathrm{diag}\left[\mathcal{E}(\tau), \mathcal{P}_{\|}(\tau), \mathcal{P}_{\perp}(\tau), \mathcal{P}_{\perp}(\tau)\right]$$

- Conditions: $\nabla_\mu T^{\mu\nu} = 0$ and $T^\mu_\mu = 0$ imply

$$\mathcal{P}_{\|} = -\mathcal{E} - \tau\dot{\mathcal{E}}\,, \qquad \mathcal{P}_{\perp} = \mathcal{E} + \frac{1}{2}\tau\dot{\mathcal{E}}$$

- Evolution of the system is captured by a single function $\mathcal{E}(\tau)$.
- Strict for an infinite energy collision of infinitely large nuclei.

[Bjorken Phys. Rev. D **27**, 140 (1983)]

Effective Temperature

- Late time: 2^{nd} order hydrodynamic expansion

$$\mathcal{E}(\tau) = \frac{3\pi^4\Lambda^4}{4(\Lambda\tau)^{4/3}}\left(1 - \frac{2c_1}{(\Lambda\tau)^{2/3}} + \frac{c_2}{(\Lambda\tau)^{4/3}} + \dots\right),$$

$$\mathcal{P}_\perp(\tau) = \frac{\pi^4\Lambda^4}{4(\Lambda\tau)^{4/3}}\left(1 - \frac{c_2}{(3\Lambda\tau)^{4/3}} + \dots\right),$$

$$\mathcal{P}_\parallel(\tau) = \frac{\pi^4\Lambda^4}{4(\Lambda\tau)^{4/3}}\left(1 - \frac{2c_1}{(\Lambda\tau)^{2/3}} + \frac{5c_2}{(3\Lambda\tau)^{4/3}} + \dots\right).$$

[Baier, Romatschke, Son, Starinets, Stephanov [0712.2451]]

- Leading order gives the Bjorken solution $\mathcal{E}_0(\tau) = \frac{3\pi^4\Lambda^4}{4(\Lambda\tau)^{4/3}}$ of ideal hydro.
- Viscous correction coefficients can be computed in holography

$$\mathcal{N} = 4\,\mathrm{SYM}: \quad c_1 = \frac{1}{3\pi}, \quad c_2 = \frac{2+\ln 2}{18\pi^2}.$$

- The energy scale Λ is the only trace of initial conditions.

Pressure Anisotropy

- Energy density defines local effective temperature

$$\mathcal{E}(\tau) = \frac{3}{4}\pi^4 T(\tau)^4 \,.$$

- Introduce the dimensionless time variable $w = \tau T(\tau)$.
- The pressure anisotropy is defined as

$$\mathcal{A}(w) = \frac{\mathcal{P}_{\|}(w) - \mathcal{P}_{\perp}(w)}{\mathcal{P}(w)} \,,$$

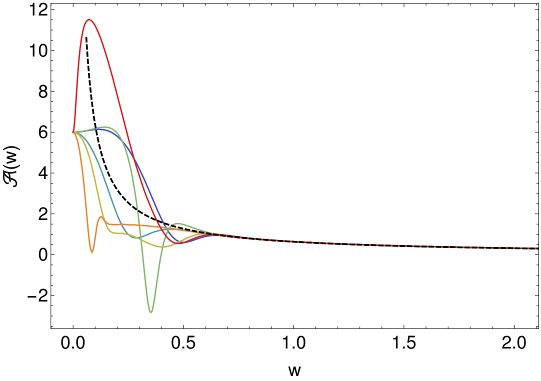
  where $\mathcal{P} = \mathcal{E}/3$ is the equilibrium pressure.
- Since $\mathcal{A} = 0$ for equilibrium, the pressure anisotropy is a measure of the distance from local equilibrium.

Universality

- **Universal approach to equilibrium:** initial state information is dissipated exponentially at early times.

Thermalization $\neq$ Hydrodynamization

$$\mathcal{A}(w_0) = \frac{\mathcal{P}_\parallel - \mathcal{P}_\perp}{\mathcal{P}} \sim 1.3 \text{ at } w_0 \sim 0.7$$



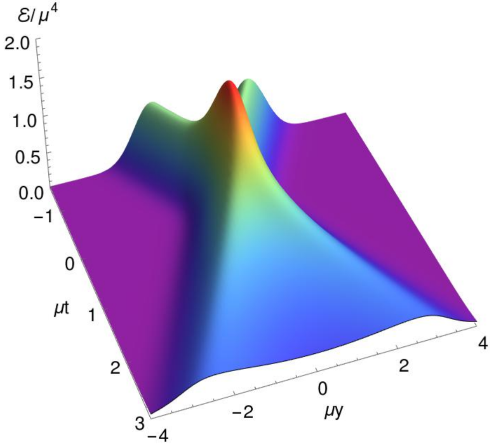
[Jankowski, Plewa, Spalinski [1411.1969]]

Shockwave Collisions in AdS

Collisions of gravitational waves in AdS as toy model for HICs.

$$ds^2 = -A(r,v,y)dv^2 + 2dv(dr + F(r,v,y)dy) + \Sigma(r,v,y)^2(e^{-2B(r,v,y)}dy^2 + e^{B(r,v,y)}d\vec{x}^2)$$

[Chesler, Yaffe [1011.3562]]



Initial Conditions

- The pre-collision geometry describing two shocks moving in $\pm\tilde{y}$-direction in Fefferman-Graham coordinates $(\tilde{r}, \tilde{t}, \tilde{y}, \vec{\tilde{r}})$ can be written down explicitly

$$ds^2 = \tilde{r}^2 \eta_{\nu\mu} d\tilde{x}^\mu d\tilde{x}^\nu + \frac{1}{\tilde{r}^2}\left(d\tilde{r}^2 + h(\tilde{t}+\tilde{y})(d\tilde{t}+d\tilde{y})^2 + h(\tilde{t}-\tilde{y})(d\tilde{t}-d\tilde{y})^2\right) .$$

- The function $h(\tilde{t}\pm\tilde{y})$ is an arbitrary function usually chosen as Gaussian

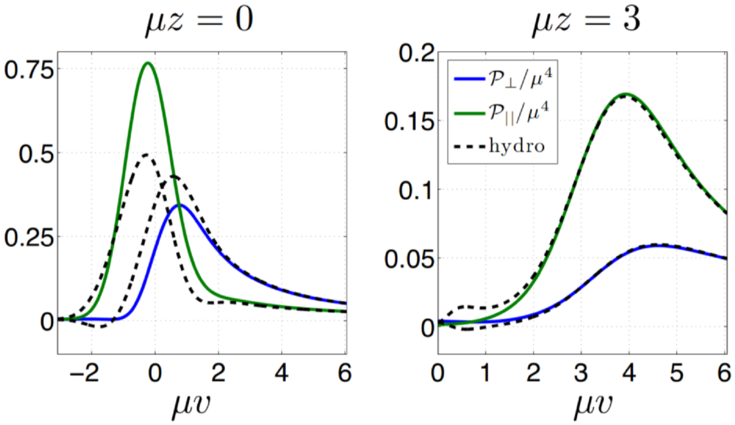
$$h(\tilde{t}\pm\tilde{y}) = \frac{\mu^3}{\sqrt{2\pi\omega^2}} e^{-\frac{(\tilde{t}\pm\tilde{y})^2}{2\omega^2}} .$$

- In this gauge the EMT describes two lumps of energy with maximum overlap at $\tilde{t} = 0$

$$\tilde{T}^{\tilde{t}\tilde{t}} = \tilde{T}^{\tilde{y}\tilde{y}} = h(\tilde{t}-\tilde{y}) + h(\tilde{t}+\tilde{y}) , \qquad \tilde{T}^{\tilde{t}\tilde{y}} = h(\tilde{t}-\tilde{y}) - h(\tilde{t}+\tilde{y}) .$$

- For the time evolution these initial conditions need to be (numerically) transformed to Eddington-Finkelstein gauge.

Hydrodynamization of Shocks

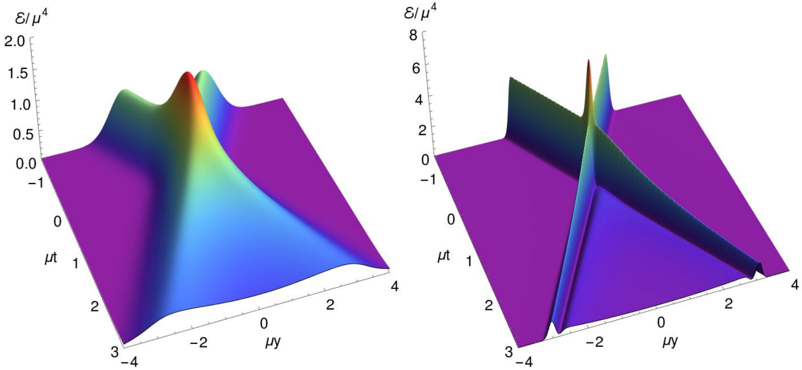


Wide vs. Narrow Shocks

Two qualitatively different dynamical regimes:

- **Wide shocks:** "full stopping", immediate hydrodynamic explosion after the collision, similar to low energy collisions at RHIC.
- **Narrow shocks:** "transparency", shocks pass through unperturbed, delayed plasma formation, similar to high energy collisions at LHC.

[Casalderrey-Solana, Heller, Mateos, van der Schee [1705.01556]]

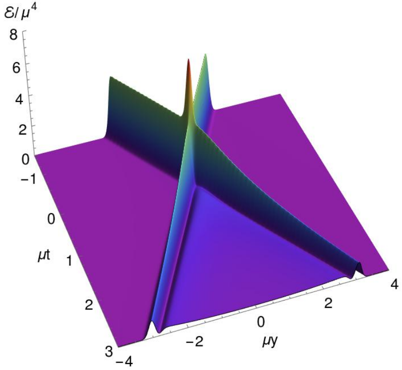


Null Energy Condition in Shckwave Collisions

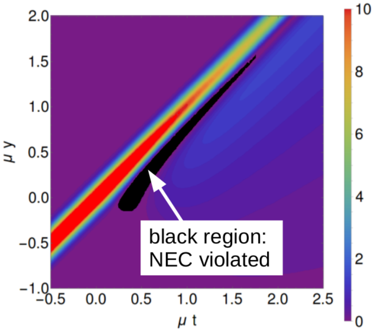
Narrow shock wave collisions can violate the null energy condition (NEC)

[Arnold, Romatschke, van der Schee, [1408.2518]]

$$T_{\mu\nu} k^{\mu} k^{\nu} \geqslant 0 \,, \quad \forall k_{\mu} k^{\mu} = 0$$



black region: $T_{\mu\nu} k^{\mu} k^{\nu} < 0$

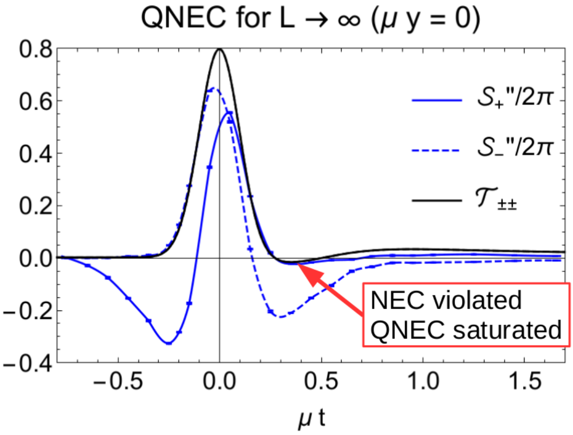


QNEC in Shockwave Collisions

In quantum field theory the quantum null energy condition (QNEC) replaces the classical NEC.

[Bousso, Fisher, Koeller, Leichenauer, Wall [1509.02542]]

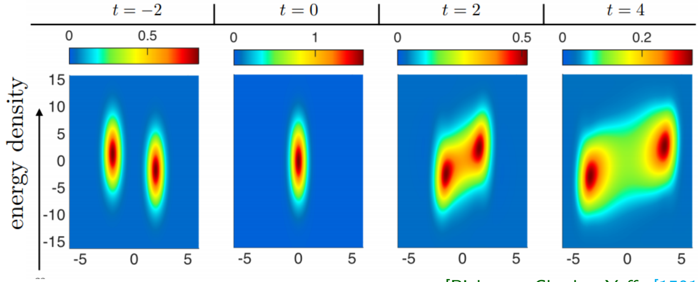
$$\langle T_{\mu\nu} k^\mu k^\nu \rangle \geqslant \frac{\hbar}{2\pi\sqrt{h}} S'' \,, \quad \forall k_\mu k^\mu = 0$$



[CE, Grumiller, van der Schee, Stanzer, [1710.09837]]

Generalizations

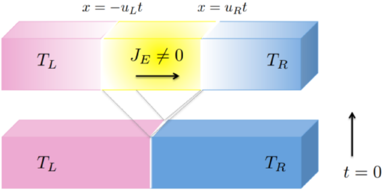
- Fully localized shocks including longitudinal and transverse dynamics.
  [Chesler, Yaffe [1501.04644]]
- Conformal symmetry breaking with non-trivial scalar field potential in the bulk.
  [Attems, Casalderrey-Solana, Mateos, Santos-Olivan, Sopuerta, Triana, Zilhao [1604.06439]]
- Finite coupling corrections using Gauss-Bonnet gravity.
  [Grozdanov, van der Schee [1610.08976]]
- Inclusion of "quark" chemical potential by adding gauge field in the bulk.
  [Folkestad, Grozdanov, Rajagopal, van der Schee,[1907.13134]]

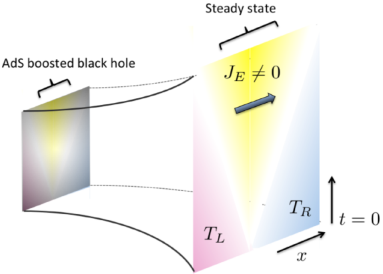


[Pichture: Chesler, Yaffe [1501.04644]]

Steady State Formation

- Thermal contact between strongly coupled quantum critical systems gives rise to a homogeneous steady state with non-vanishing energy flow.
- D=2: Steady state is described by Lorentz boosted equilibrium state with $T = \sqrt{T_L * T_R}$.
- Dynamics completely fixed by conformal symmetry: two shock waves with constant profile moving at the speed of light to cold and warm side.

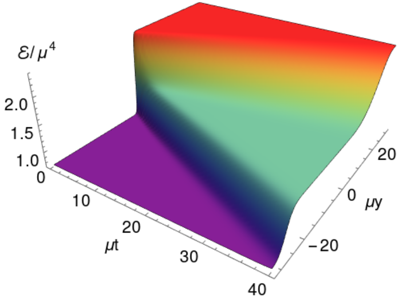


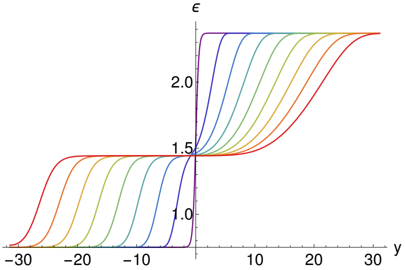


[Bhaseen, Doyon, Lucas, Schalm [1311.3655]]

Steady State Formation in AdS_5

- $D > 2$: Hydrodynamic solution not unique. Two shocks mathematically correct solution to ideal hydrodynamics.
- Problem: shockwave propagating to the warm bath violates (locally) the second law of thermodynamics.
- Physical solution: shockwave moving to cold, rarefaction wave moving to warm.
- Holography automatically delivers physical solution.





Summary of Lecture 2

- Spectral methods are an efficient tool to numerically solve differential equations.
- The late time dynamics of the homogeneous and anisotropic plasma is described by exponentially damped oscillations (QNMs).
- Boost invariant model evolves towards universal hydrodynamic regime.
- Collisisons of gravitational shocks in AdS as model for HICs.
- In Tutorial 2 in the afternoon we will learn how to solve BVPs using spectral methods. Furthermore, we will solve compute the time evolution and analyze the late time behaviour.
- Steady state formation with end state has non-trivial energy flow.
- Recent developments include localized shocks, conformal symmetry breaking and finite coupling corrections.

Outline of Lecture 3

1. Entanglement Entropy
2. Holographic Entanglement Entropy
3. Tutorial 3: Shoot & Relax in AdS Space

1. Entanglement Entropy

Why is entanglement entropy interesting?

- Entanglement entropy is a measure for entanglement in quantum systems.
- A concept that originated in quantum information theory.
  [Nielsen, Chuang Cambridge University Press, 2010]
- It was proposed as a way of understanding black hole entropy.
  [Bombelli, Koul, Lee, Sorkin 86, Srednicki [hep-th/9303048]]
- It provides a measure for dofs in renormalization group flows.
  [Casini, Huerta [cond-mat/0610375],[1202.5650]]
- Order parameter for exotic phase transitions in quantum critical systems.
  [Osborne, Nielsen [quant-ph/0202162], Vidal,Latorre, Rico, Kitaev [quant-ph/0211074]]
- In holography the Ryu-Takayanagi formula relates the area of extremal surfaces in gravity theory to quantum entanglement.
  [Ryu, Takayanagi [hep-th/0603001],[hep-th/0605073]]

$$S_A = \frac{\mathcal{A}}{4G_N}$$

- Recently used to formulate the quantum null energy condition (QNEC), a universal energy bound in QFTs.
  [Bousso, Fisher, Leichenauer, Wall [1506.02669], Bousso, Fisher, Koeller, Leichenauer, Wall [1509.02542]]

$$\langle T_{\mu\nu}(x) k^\mu k^\nu \rangle \geqslant S''_A(x)\,, \quad \forall k^2 = 0$$

Entanglement Entropy

- Divide system into two parts $(A, B = \bar{A})$
- Assume that the Hilbert space factorizes[1]

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

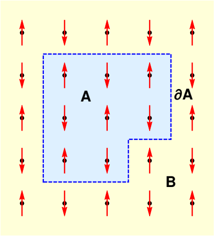
- Compute reduced density matrix by tracing over $\mathcal{H}_B$

$$\rho_A = \mathrm{Tr}_B\, \rho$$

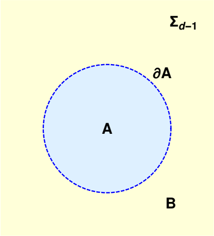
- Entanglement entropy is defined as the von Neumann entropy of ρ_A

$$S_A = -\mathrm{Tr}_A\, \rho_A \log \rho_A$$

Lattice Theory



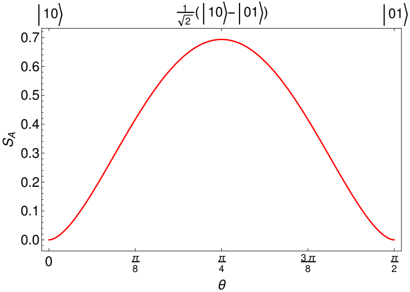
Quantum Field Theory



[1] This assumption can be problematic for instance in lattice gauge theories, where the gauge invariant variables are non-local.

Simple quantum mechanical two spin system

- A quantum system with two particles (A,B) of spin 1/2 has per construction
$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$
- Consider a general superposition state of two product states
$$|\psi\rangle = \cos\theta|01\rangle + \sin\theta|10\rangle, \quad |ij\rangle = |i\rangle_A \otimes |j\rangle_B \quad \forall i,j = 0,1$$
- It is an simple quantum mechanics exercise to compute the entanglement entropy
$$\rho = |\psi\rangle\langle\psi| \rightarrow \rho_A = \sum_i {}_B\langle i|\rho|i\rangle_B \rightarrow S_A = \sum_i {}_A\langle i|\rho_A \log \rho_A|i\rangle_A$$
$$S_A(\theta) = -\cos^2(\theta)\log(\cos^2\theta) - \sin^2(\theta)\log(\sin^2\theta)$$
- For $\theta = \pi/4$ one obtains a maximally entangled state $S_A = \log \dim \mathcal{H}_A = \log 2$. Such states are called Bell states or Einstein-Podolski-Rosen (EPR) pairs.



Entanglement Entropy in QFT

- The major contribution to entanglement entropy in QFT comes from Einstein-Podolski-Rosen (EPR) pairs across the entangling surface ∂A.
- EE is divergent in QFT: ∂A is continuous $\Rightarrow$ infinitely many EPR pairs.

Results for d-dim. free field theories:

- **Universal area-law UV-scaling:** (char. length: $\ell \ll 1$, UV-cutoff: $\epsilon \ll \ell$)

$$S_A = s_{d-2}\left(\frac{\ell}{\epsilon}\right)^{d-2} + s_{d-4}\left(\frac{\ell}{\epsilon}\right)^{d-4} + \ldots + (-1)^{\frac{d-2}{2}} s_0 \log\frac{\ell}{\epsilon} + \mathcal{O}(\epsilon), \quad d \text{ even}$$

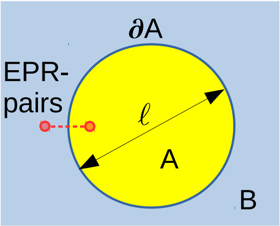
[Srednicki [hep-th/9303048]]

- **Non-universal IR-scaling:** $\ell \gg 1$
  Ground states show area-law scaling

$$S_A \propto (\ell/\epsilon)^{d-2}$$

  Thermal states show volume-law sclaling

$$S_A \propto (\ell/\epsilon)^{d-1}$$



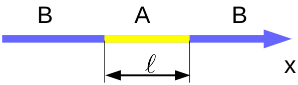
Entanglement Entropy in CFT_2

- Computing EE in interacting QFTs in $d > 2$ is usually intractable.
- The exception are CFT_2, where explicit results can be obtained via the Replica Method.

[Calabrese, Cardy [quant-ph/0505193] [0905.4013]]

- Result for CFT_2 on $\mathcal{R}^{1,1}$:

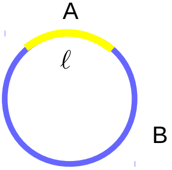
$$S_A = \frac{c}{3} \log \frac{\ell}{\epsilon} + \text{finite}\,.$$



- CFT_2 on $\mathcal{R} \times S_1$ has two interpretations:

1) CFT on a compact space of size ℓ_{S_1}

$$S_A = \frac{c}{3} \log \left(\frac{\ell_{S_1}}{\pi \epsilon} \sinh \frac{\ell}{\ell_{S_1}} \right)\,.$$



2) Euclidean CFT with $T^{-1} = \beta = \ell_{S_1}$

$$S_A = \frac{c}{3} \log \left(\frac{\beta}{\pi \epsilon} \sinh \frac{\pi \ell}{\beta} \right)\,.$$

Time Dependent States: Quantum Quench in CFT_2

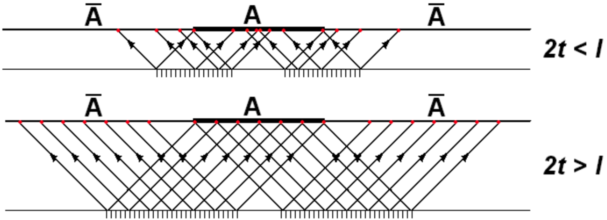
- Assume $|\psi_0\rangle$ to be the ground state of the Hamiltonian H_0.
- Suddenly change (quench) the Hamiltonian $H_0 \to H$ at $t = 0$.
- $|\psi_0\rangle$ becomes excited state of H with unitary time evolution

$$\rho(t) = e^{-iHt}|\psi_0\rangle\langle\psi_0|e^{iHt}$$

- For the entanglement entropy one finds

$$S_A(t) = \frac{c}{3}\log\frac{\beta}{\epsilon} + \Delta S_A(t)\,, \quad \Delta S_A(t) \propto \begin{cases} \frac{\pi c}{6\epsilon}t & \text{if } t < \frac{l}{2} \\ \frac{\pi c}{12\epsilon}l & \text{if } t \geqslant \frac{l}{2} \end{cases}$$

- Quasi-particle picture: EPR pairs created at $t = 0$, propagate with speed of light in CFT_2, only pairs contribute where one particle is within A.



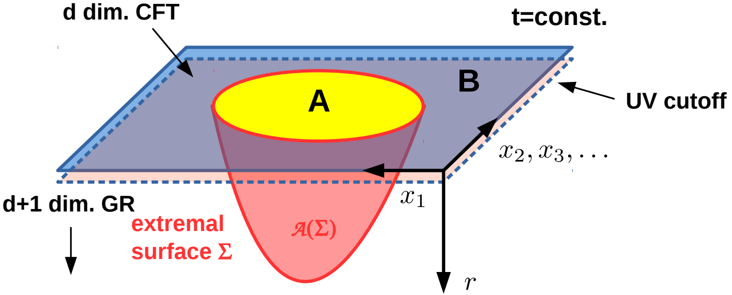
[Picture: Abajo-Arrastia, Aparício, López [hep-th/1006.4090]]

2. Holographic Entanglement Entropy

The Ryu-Takayanagi Formula

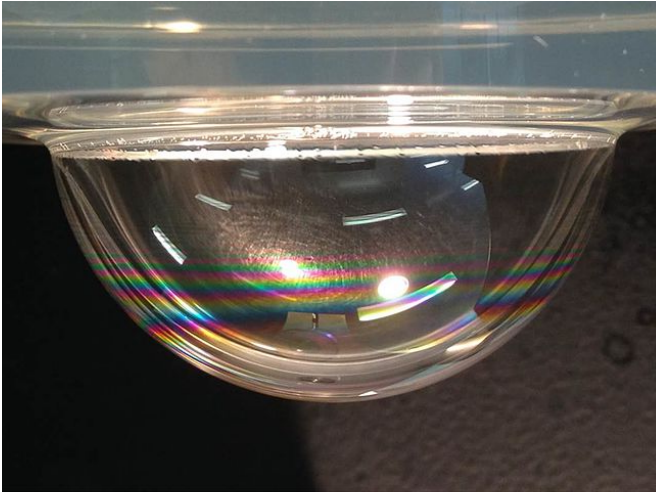
- In holography S_A can be computed from the area of extremal co-dimension 2 bulk surfaces homologous to the entangling region A.
[Ryu, Takayanagi [hep-th/0603001], Hubeny, Rangamani, Takayanagi [0705.0016]]
- Extremal surfaces extremize the area functional in AdS space.
- Infinite extension to the boundary corresponds to UV-divergence in CFT
$\Rightarrow$ Regularize the area by chopping the surface at finite r.

$$S_A = \frac{\mathcal{A}_A}{4G_N}$$



The Ryu-Takayanagi Formula

$$S_A = \frac{\mathcal{A}_A}{4G_N}$$



[Picture: Wikipedia]

Entanglement Inequalities

- Entanglement entropy satisfies a number of inequalities that are hard to proof in QFTs. [Nielsen, Chuang 2000]
- Bipartite systems $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ satisfy subadditivity (SA)

$$S_A + S_B \geqslant S_{A \cup B}$$

  SA motivates mutual information which is per construction finite

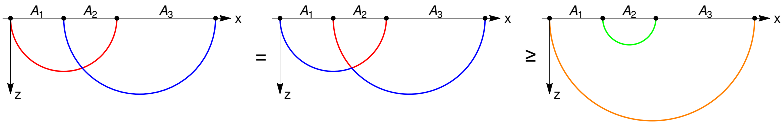
$$I_{AB} = S_A + S_B - S_{A \cup B}$$

- Tripartite systems $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ satisfy strong subadditivity (SSA)

$$S_{A \cup B} + S_{B \cup C} \geqslant S_{A \cup B \cup C} + S_B$$

- Proofing SSA in holography is simple:

$$S_{1,2} + S_{2,3} \geqslant S_{1,2,3} + S_2$$



Extremal Surface Equations in AdS

We start with the line element of a general asymptotic AdS_{d+1} spacetime

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu$$

The embedding $X^\mu = X^\mu(\sigma^a, z)$ of a co-dimension 2 surface is parametrized with $d-2$ intrinsic coordinates σ^a and the bulk coordinate z.
The area functional can be written in terms of the induced metric $H_{\alpha\beta}$

$$\mathcal{A} = \int dz d^{d-2}\sigma \sqrt{H[X]}, \quad H_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}$$

Variation of the area functional $\delta\mathcal{A} = 0$ with respect to the embedding functions gives the differential equation for the surface

$$\frac{1}{\sqrt{H}} \partial_\alpha (\sqrt{H} H^{\alpha\beta} \partial_\beta X^\mu) + H^{\alpha\beta} \partial_\alpha X^\sigma \partial_\beta X^\nu \Gamma^\mu_{\sigma\nu} = 0$$

Solving this non-linear PDE subject to BCs describing the entangling region is hard. Explicit solutions are only available for highly symmetric cases in which the entangling region respects the symmetries of the bulk geometry.

Reduction to a Geodesic Problem

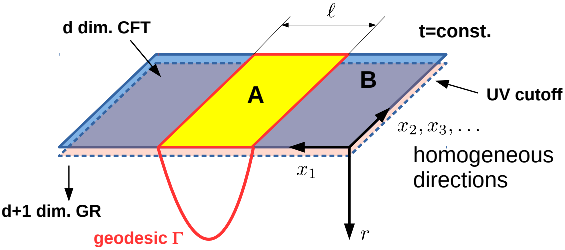
In the AdS_3 case the minimal surfaces reduce to geodesics determined by the geodesic equation

$$\ddot{X}^\mu + \Gamma^\mu_{\rho\sigma}\dot{X}^\rho\dot{X}^\sigma = 0$$

A similar reduction also works in higher dimensions, if the entangling region does not break the symmetries of the bulk geometry, e.g. stripe regions.

$$\mathcal{A} = \int d^3\sigma\sqrt{\det\left(g_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma^a}\frac{\partial X^\nu}{\partial\sigma^b}\right)} = \underbrace{\int dx_3 \int dx_2}_{\infty \text{ volume factor}} \int d\sigma\sqrt{\Omega^2 g_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma}\frac{\partial X^\nu}{\partial\sigma}}$$

The Christoffel symbols $\Gamma^\mu_{\rho\sigma}$ are then computed from the metric with an additional conformal factor $\tilde{g}_{\mu\nu} = \Omega(z,t,x_1)^2 g_{\mu\nu}$.



Minimal Surfaces in BTZ Geometry

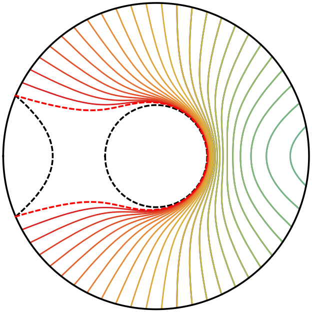
BTZ geometries are holographic duals of thermal states with $T = \beta^{-1} = \frac{1}{2\pi} r_+$ in CFT_2 on $\mathrm{S}^1 \times \mathbb{R}$.

$$ds^2 = -(r^2 - r_+^2)dt^2 + \frac{dr^2}{r^2 - r_+^2} + r^2 d\varphi^2 \,, \quad \varphi \in [0, 2\pi)$$

For the entangling region $A = \{t = t_0, -\varphi_0 < \varphi < \varphi_0\}$ the one obtains

$$S_A = \frac{\mathcal{A}}{4G_N^{(3)}} = \frac{c\mathcal{A}}{6} = \begin{cases} \frac{c}{3} \log\left(\frac{\beta}{\pi\epsilon} \sinh\left(\frac{R}{\beta}\varphi_0\right)\right) & \varphi_0 < \varphi_* \,, \\ \frac{c}{3}\pi r_+ + \frac{c}{3} \log\left(\frac{\beta}{\pi\epsilon} \sinh\left(\frac{R}{\beta}(\pi - \varphi_0)\right)\right) & \varphi_0 \geqslant \varphi_* \,, \end{cases}$$

- **Homology constraint:** Surfaces must be smoothly retractable to A.
- $\varphi = \varphi_*$ (red dashed): saddle points of the area functional exchange dominance[1].
- $\varphi > \varphi_*$ (black dashed): two disconnected surfaces $\Rightarrow S_A = S_{BH} + S_{\bar{A}}$.
- **Entanglement plateau:** S_A saturates to thermal entropy for large sub-regions.
  [Hubeny, Maxfield, Rangamani, Tonni [1306.4004]]



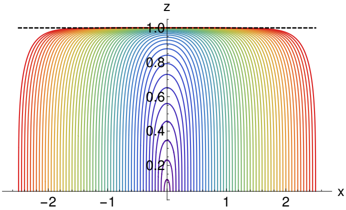
[1] The case $\varphi = \varphi_*$ saturates the Araki-Lieb inequality $|S_A - S_B| \leqslant S_{A\cup B}$.

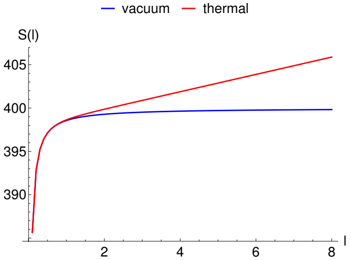
Schwarzschild Black Brane

AdS_{d+1} black brane geometries are holographic duals of thermal states with $T = \frac{d}{4\pi}\sqrt[d]{M}$ in CFT_4 on $\mathcal{M}_d$.

$$ds^2 = \frac{1}{z^2}\left(-(1 - Mz^d)dt^2 - 2dzdt + d\vec{x}^2\right)$$

- surfaces remain always outside the BH horizon.
- We recover the universal area law for small regions ($\ell < z_h$).
- For large regions ($\ell > z_h$) the vacuum ($M = 0$) gives the area law and thermal states ($M \neq 0$) the volume law[5].





[5]The plot is for the entanglement densities in the 1-dim subspace of the stripe region, i.e. the are is 0-dimensional and the Volume is 1-dimensional.

Vaidya Quench

- Vaidya-AdS$_{d+1}$ geometry in Eddington-Finkelstein coordinates

$$ds^2 = \frac{1}{z^2}\left(-\left(1 - M(t)z^d\right)dt^2 - 2dzdt + d\vec{x}^2\right)\ , M(t) = \frac{1}{2}\left(1 + \tanh(at)\right)$$

- Quench: Infalling matter shell = sudden injection of energy in CFT
- **Entanglement Tsunami:**

$t \leqslant \beta_{\text{eq}}: \quad \Delta S_A(t) = \frac{\pi}{d-1}\epsilon \text{Area}(\partial A)t^2 + \ldots \quad$, where $\beta_{\text{eq}} = T_{\text{eq}}^{-1}$

$\beta_{\text{eq}} \ll t \ll \ell/2: \quad \Delta S_A(t) = v_E s_{\text{eq}} \text{Area}(\partial A)t + \ldots$

$t \gg \ell/2: \quad S_A(t) = s_{\text{eq}} \text{Vol}(\partial A)$

[Liu, Suh [1305.7244]]