Quantum Null Energy Condition and its (non)saturation in 2d CFTs

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Introduction

Energy Conditions

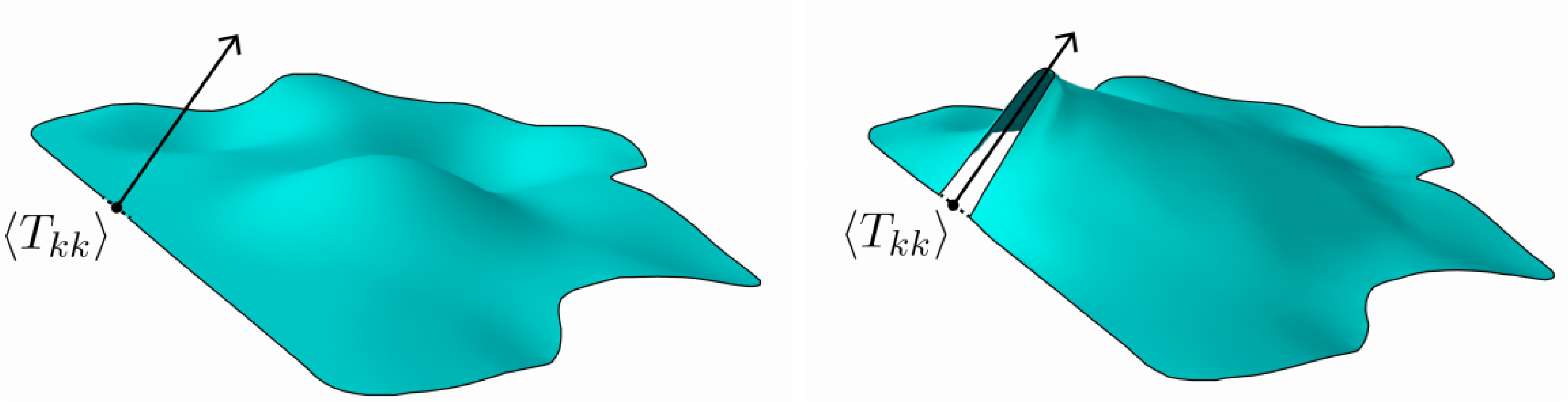
Constraints on the energy-momentum tensor (EMT) which are necessary to proof certain theorems in general relativity. (area theorem, singularity theorems, ...)

EC	Inequality	Local?	True?	Violated by
SEC	$T_{\mu\nu}t^\mu t^\nu \geq T^\mu_\mu t^\nu t_\nu$	yes	NO!	free scalar field
WEC	$T_{\mu\nu}t^\mu t^\nu \geq 0$	yes	NO	negative cosmological constant
NEC	$T_{\mu\nu}k^\mu k^\nu \geq 0$	yes	no	quantum effects
ANEC	$\int_\gamma T_{\mu\nu}k^\mu k^\nu \geq 0$	no	yes	not violated in reasonable QFTs
QNEC	$\langle T_{kk}\rangle \geq \frac{\hbar}{2\pi\sqrt{h}}S''$	yes	yes	not violated in reasonable QFTs

QNEC is the only known local energy condition which holds in any QFT.

Quantum Null Energy Condition

$$\boxed{\langle T_{ab}k^a k^b\rangle \geq \frac{\hbar}{2\pi\sqrt{h}} S'' \quad \forall k^2 = 0}$$



[picture source: Bousso-Fisher-Leichenauer-Wall 1512.06109]

Note that S'' can have either sign:

- $S'' > 0$ QNEC **stronger** than NEC
- $S'' = 0$ QNEC **equivalent** to NEC
- $S'' < 0$ QNEC **weaker** than NEC

Current Status of QNEC in d>2

There exist several proofs:

- Free bosonic field theories [Bousso, Fisher, Koeller, Leichenauer, and Wall 1509.02542]
- Holographic field theories [Koeller and Leichenauer 1512.06109]
- General QFTs [Balakrishnan, Faulkner, Khandker and Wang 1706.09432]
- In d>2 QNEC was conjectured to saturate for all states [Leichenauer, Levine, Shahbazi and Moghaddam 1802.02584]

QNEC in CFT2

$$\boxed{\langle T_{kk} \rangle \geq \frac{\hbar}{2\pi}\left(S'' + \frac{6}{c}(S')^2\right) \qquad \forall k^2 = 0}$$

- In two dimensions QNEC is stronger than in higher dimensions.
- In d=2 QNEC does not need to saturate. [Khandker-Kundu-Li 1803.03997]
- Interesting question: Under which condition does it/does it not saturate?
- We will study this question using holography.

QNEC saturation for all Bañados geometries

Bañados geometries

We consider a CFT2 on a cylinder in the large central charge limit

$$ds^2 = -dt^2 + d\varphi^2 = -dx^+ dx^- \qquad x^\pm = t \pm \varphi \qquad c = \frac{3}{2G_N} \gg 1$$

States dual to Bañados geometries:

$$ds^2 = \frac{dz^2 - dx^+ dx^-}{z^2} + \mathcal{L}^+(x^+)(dx^+)^2 + \mathcal{L}^-(x^-)(dx^-)^2 - z^2 \mathcal{L}^+(x^+)\mathcal{L}^-(x^-) dx^+ dx^-$$

Most general solution of 3D vacuum Einstein gravity with AdS boundary conditions.

Poincaré patch AdS: $\mathcal{L}^\pm = 0$ global AdS: $\mathcal{L}^\pm = -\frac{1}{4}$ BTZ: $\mathcal{L}^\pm = \mathrm{const.}$

AdS/CFT relates Bañados geometries to excited CFT2 states $|\mathcal{L}^+, \mathcal{L}^-\rangle$

$$2\pi \langle \mathcal{L}^+, \mathcal{L}^- | T_{\pm\pm}(x^\pm) | \mathcal{L}^+, \mathcal{L}^- \rangle = \frac{6}{c} \mathcal{L}^\pm(x^\pm)$$

Uniformization and Hill's equation

All Bañados metrics are locally Poincaré patch AdS3

$$x_P^{\pm} = \int \frac{dx^{\pm}}{\psi^{\pm 2}} - \frac{z^2 \psi^{\mp\prime}}{\psi^{\pm 2}\, \psi^{\mp}(1 - z^2/z_h^2)} \quad z_P = \frac{z}{\psi^{+}\psi^{-}(1 - z^2/z_h^2)} \quad z_h^2 = \frac{\psi_a^{+}\psi_b^{-}}{\psi_a^{+\prime}\psi_b^{-\prime}}$$

The functions $\psi^{\pm}$ are solutions to Hill's equation

$$\psi^{\pm\prime\prime} - \mathcal{L}^{\pm}\psi^{\pm} = 0$$

The two independent solutions are normalized to unit Wronskian

$$\psi_1^{\pm}\psi_2^{\pm\prime} - \psi_2^{\pm}\psi_1^{\pm\prime} = \pm 1$$

Holographic entanglement entropy in AdS3/CFT2

HEE factorizes into a sum of holomorphic and anti-holomorphic contributions

[Sheikh-Jabbari and Yavartanoo 1605.00341]

$$S = S^+ + S^- \qquad\qquad S^\pm = \frac{c}{6}\ln(\ell^\pm(x_1^\pm, x_2^\pm))$$

$$\ell^\pm(x_1^\pm, x_2^\pm) = \psi_1^\pm(x_1^\pm)\psi_2^\pm(x_2^\pm) - \psi_2^\pm(x_1^\pm)\psi_1^\pm(x_2^\pm)$$

To compute entanglement entropy one has to solve Hill's equation for a given function $\mathcal{L}^\pm(x^\pm)$ that describes the state and boundary conditions for $\psi_{1,2}^\pm$ encoding the entangling region.

Example: Entanglement entropy of the Poincaré patch vacuum $\mathcal{L}^\pm = 0$

$$\psi_1^+ = x^+ \qquad \psi_2^+ = 1 = \psi_1^- \qquad \psi_1^+ = x^+ \qquad \ell = |x_1^+ - x_2^+| = |x_1^- - x_2^-|$$

$$S = \frac{c}{3}\ln\frac{\ell}{\epsilon}$$

Bañados geometries saturate QNEC

After defining

$$V = \exp\left(\tfrac{6}{c} S\right) = \frac{\ell^+(x_1^+, x_2^+)}{\epsilon^2}$$

and using the fact that $\psi^{\pm}_{1,2}$ solve Hill's equation which gives

$$\frac{V''}{V} = \frac{6}{c}\left(S'' + \frac{6}{c}(S')^2\right) = \mathcal{L}^+$$

we find that QNEC saturates for all states dual to Bañados geometries

$$\frac{c}{6}\mathcal{L}^+ = S'' + \frac{6}{c}(S')^2 = 2\pi\langle T_{++}\rangle$$

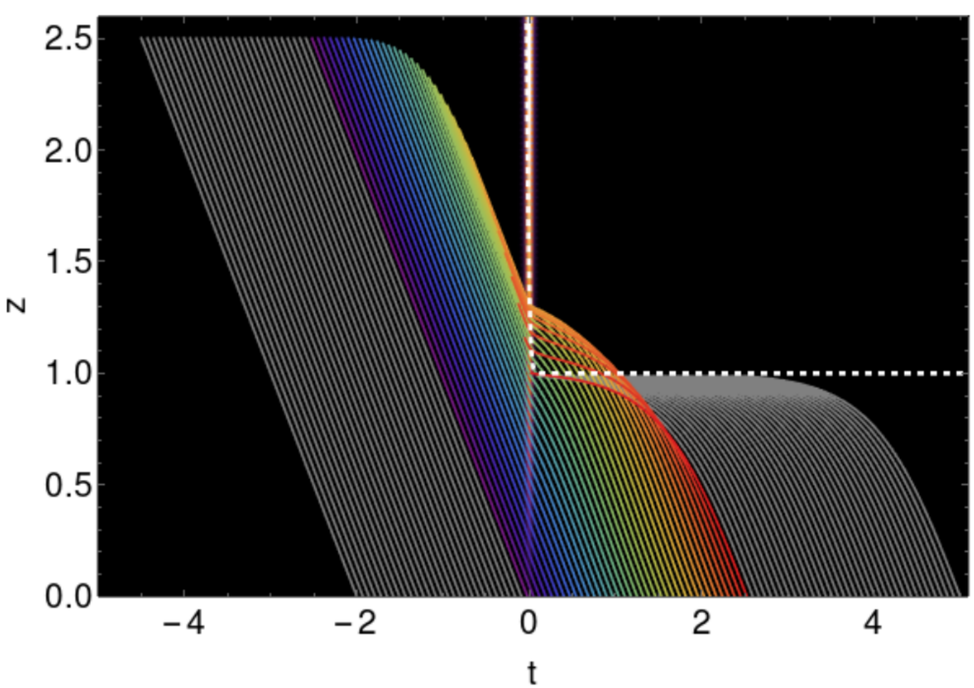
QNEC non-saturation in presence of bulk matter

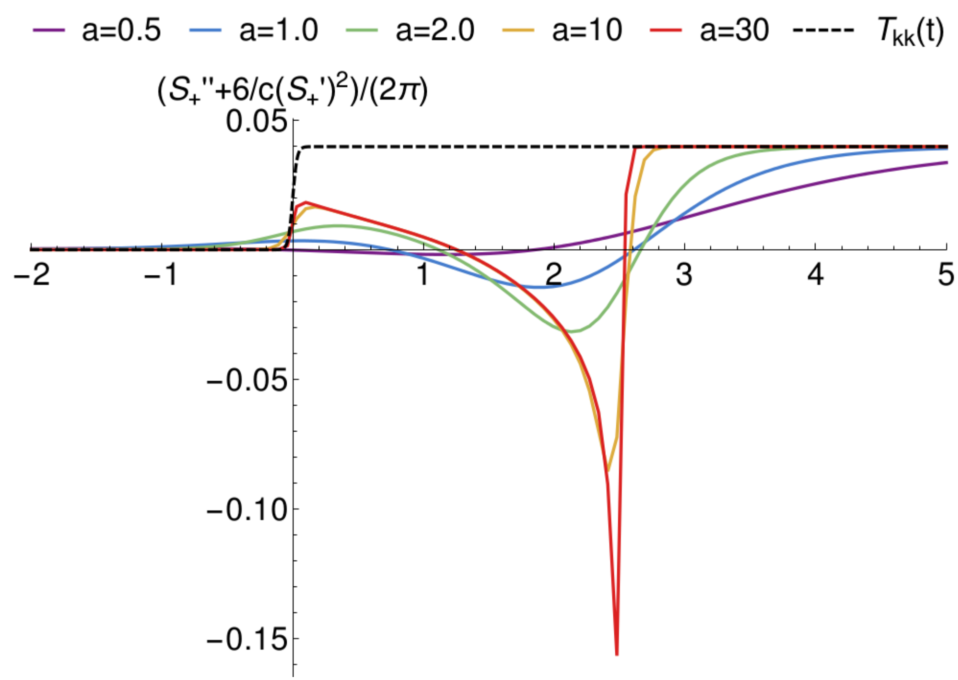
QNEC in a globally quenched state

Holographic model: AdS-Vaidya geometry

$$ds^2 = \frac{1}{z^2}(-(1-M(t)z^2)dt^2 - 2dzdt + dx^2), \quad M(t) = \frac{1}{2}(1+\tanh(at))$$

Global quench from vacuum to a thermal state of temperature $T = \frac{1}{2\pi}\sqrt{M(\infty)}$





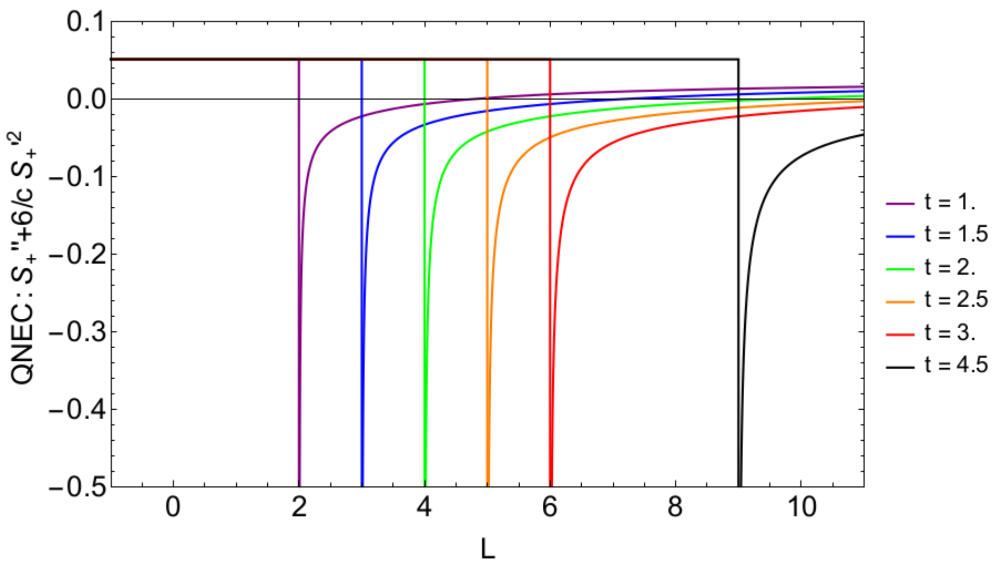
QNEC does **not saturate** if the **RT-surface crosses bulk matter** $T^{bulk}_{tt}(z,t) = \frac{z}{2}M'(t)$

Divergence and half-saturation

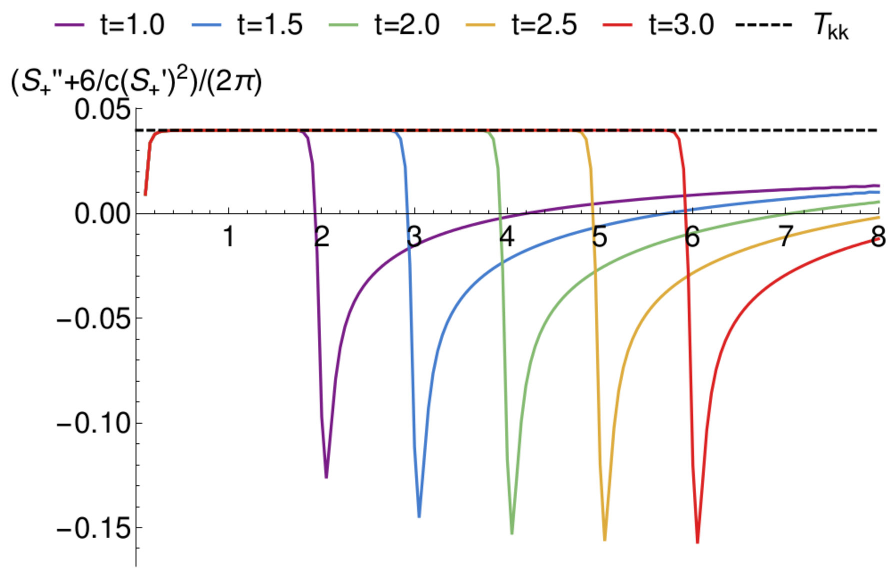
Perturbative calculation with $M(t) = \epsilon\theta(t)$ for $\epsilon \ll 1$ gives

$$\frac{S'' + \frac{6}{c}(S')^2}{\langle T_{kk}\rangle} = 1 + \frac{t_0(l-t_0)\sqrt{l^2-4t_0^2}}{l^3} - \frac{\sqrt{l+2t_0}}{2\sqrt{l-2t_0}}$$

- QNEC diverges when geodesic dip crosses the matter shell at $l = 2t_0$
- QNEC half-saturates for large separation: $\lim_{l \gg t_0}\left(\frac{S''_\pm + \frac{6}{c}(S'_\pm)^2}{\langle T_{\pm\pm}\rangle}\right) = \frac{1}{2} \pm \frac{t_0}{l} + \mathcal{O}(t_0^2/l^2)$



Perturbative solution



Full numeric solution

Finite-c corrections to QNEC

Quantum corrections to HEE

$$S = \frac{\mathcal{A}}{4G_N} + \frac{\delta\mathcal{A}}{4G_N} + S_{bulk}$$

[Faulkner, Lewkowycz, and Maldacena 1307.2892]

$\mathcal{A}$ … usual area contribution form the Ryu-Takayanagi surface

$\delta\mathcal{A}$ … shift in the RT-surface due to quantum backreaction

S_{bulk} … bulk entanglement entropy for region enclosed by the RT-surface

For QNEC we need to compute all these contributions as functions of a parameter parametrizing the light like deformation of the entangling region.

Quantum backreaction from bulk scalar field

Single particle quantum field in AdS3 dual to a CFT2 primary state of weight h

$$\phi = \frac{a}{\sqrt{2\pi}} \frac{e^{-2iht}}{(1+r^2)^h} + \frac{a^\dagger}{\sqrt{2\pi}} \frac{e^{2iht}}{(1+r^2)^h} \qquad m^2 = 4h(h-1) \qquad h \geq 1/2$$ (BF bound)

Bulk energy momentum tensor

$$T_{\mu\nu} =: \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu}((\nabla\phi)^2 + m^2\phi^2) :$$

Evaluated for a single particle excited state $|\psi\rangle = a^\dagger |0\rangle$

$$\langle\psi|T_{tt}(r)|\psi\rangle = \frac{2h(2h-1)}{\pi} \frac{1}{(1+r^2)^{2h-1}} \qquad \langle\psi|T_{rr}(r)|\psi\rangle = \frac{2h}{\pi} \frac{1}{(1+r^2)^{2h+1}}$$

$$\langle\psi|T_{\phi\phi}(r)|\psi\rangle = \frac{2hr^2}{\pi} \frac{(1-2h)r^2+1}{(1+r^2)^{2h+1}}$$

Quantum backreacted geometry

The leading order quantum correction to the geometry follows form solving the semi-classical Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R - g_{\mu\nu} = 8\pi G_N \langle \psi | T_{\mu\nu} | \psi \rangle$$

The geometry that solves the semi-classical Einstein equations is known

$$ds^2 = -(r^2 + G_1(r)^2) dt^2 + \frac{dr^2}{r^2 + G_2(r)^2} + r^2 d\varphi^2$$

$$G_1(r) = 1 - 8G_N h + \mathcal{O}(G_N^2) \qquad G_2(r) = 1 - 8G_N h \left(1 - \frac{1}{(r^2+1)^{2h-1}}\right) + \mathcal{O}(G_N^2)$$

The quantum corrected holographic energy momentum tensor is given by

$$2\pi \langle T_{\pm\pm} \rangle = -\frac{c}{24} + h + \mathcal{O}(h^2/c)$$

Area contribution

One has to extremize the area functional on the backreacted geometry with boundary conditions parametrizing the light-like deformation of the entangling region.

$$\mathcal{A} = \int_0^{\Delta\varphi+\lambda} d\varphi \mathcal{L}(z, \dot{z}, \dot{t}) \qquad \delta\mathcal{A} = 0$$

A long and tedious calculation gives a closed form solution for the RT-part to entanglement entropy (in agreement with [Belin, Iqbal, and Lokhande 1805.08782])

$$S_{\mathrm{RT}} = \frac{1}{4G_N}\mathcal{A}\Big|_{\lambda\to 0} = \frac{c}{3}\ln\frac{2\sin\frac{\Delta\varphi}{2}}{z_{\mathrm{cut}}} + 4h\Bigg(1 - \frac{\pi - |\pi-\Delta\varphi|}{2}\cot\frac{\pi-|\pi-\Delta\varphi|}{2}$$
$$+ \frac{\sqrt{\pi}\,\Gamma[2h]}{\Gamma[2h+\frac{3}{2}]}\sin^{4h}\frac{\Delta\varphi}{2}\left(h\,{}_2F_1(\tfrac{1}{2},1,2h+\tfrac{3}{2};-\tan^2\frac{\Delta\varphi}{2}) - h - \frac{1}{4}\right)\Bigg) + \mathcal{O}(1/c)$$

The corresponding RT-part for QNEC

$$S''_{RT} + \frac{6}{c}(S'_{RT})^2 = -\frac{c}{24} + h - \frac{h\sqrt{\pi}\Gamma[2h+2]}{4\Gamma[2h+\frac{3}{2}]}^{4h-2}\frac{\Delta\varphi}{2} + \mathcal{O}(1/c)$$

Bulk entanglement contribution

Here we restrict to small entangling regions, such that the entropy can be computed from the expectation value of the vacuum modular Hamiltonian.

[Belin, Iqbal, and Lokhande 1805.08782]

$$\Delta S_{bulk} = 2\pi \langle \psi | H_0 | \psi \rangle \qquad \rho_{vac} = e^{-2\pi H_0} \qquad \Delta\varphi \ll 1$$

The expectation value of the modular Hamiltonian can be expressed as an integral in Rindler coordinates.

$$\Delta \langle H_0 \rangle = \int_{\Sigma_A} d\Sigma_A \sqrt{|g_{\Sigma A}|} \xi^\nu n^\mu \langle \psi | T_{\mu\nu} | \psi \rangle$$

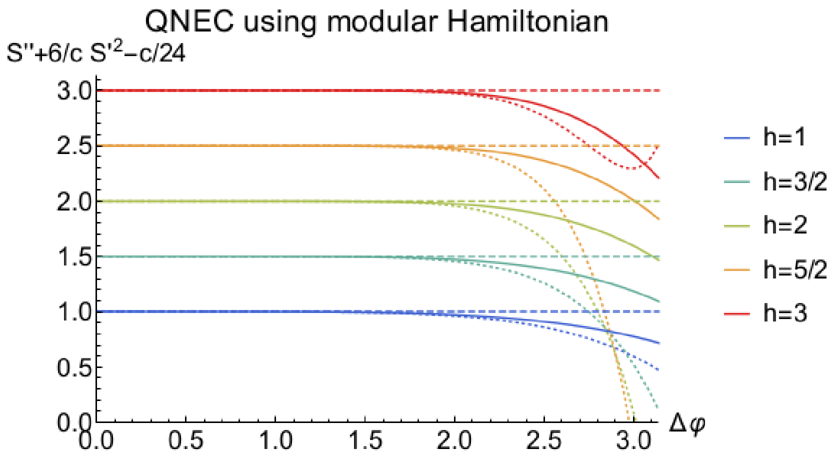
Needs to be evaluated in a boosted frame that realizes the light like shift of the entangling region. (result on next page)

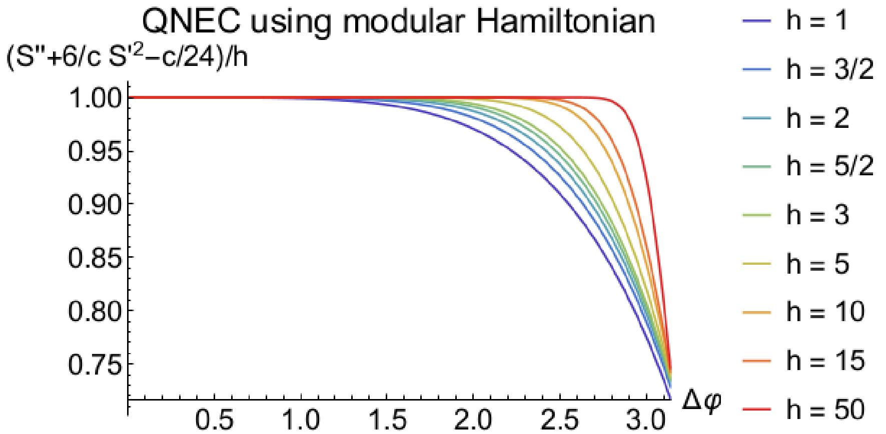
For the total (RT+bulk) contribution in the small expansion we obtain

$$2\pi \langle T_{\pm\pm} \rangle - S'' - \frac{6}{c}(S')^2 = +\mathcal{O}(\Delta\varphi^{4h})$$

QNEC for the small regions

h	$\Delta\langle H_0\rangle'' + \frac{12}{c}\Delta\langle H_0\rangle' S_0'$	$S_{\mathrm{RT}}'' + \frac{6}{c}(S_{\mathrm{RT}}')^2 - h + \frac{c}{24}$
$1/2$	$\frac{1}{3} + \mathcal{O}(\Delta\varphi^2)$	$-\frac{1}{3}$
1	$\frac{\Delta\varphi^2}{5} - \frac{43\Delta\varphi^4}{2520} + \frac{\Delta\varphi^6}{21600} + \mathcal{O}(\Delta\varphi^8)$	$-\frac{\Delta\varphi^2}{5} + \frac{\Delta\varphi^4}{60} - \frac{\Delta\varphi^6}{1800} + \mathcal{O}(\Delta\varphi^8)$
$3/2$	$\frac{3\Delta\varphi^4}{35} - \frac{73\Delta\varphi^6}{5040} + \frac{1571\Delta\varphi^8}{1663200} + \mathcal{O}(\Delta\varphi^{10})$	$-\frac{3\Delta\varphi^4}{35} + \frac{\Delta\varphi^6}{70} - \frac{3\Delta\varphi^8}{2800} + \mathcal{O}(\Delta\varphi^{10})$
2	$\frac{2\Delta\varphi^6}{63} - \frac{667\Delta\varphi^8}{83160} + \frac{2789\Delta\varphi^{10}}{3088800} + \mathcal{O}(\Delta\varphi^{12})$	$-\frac{2\Delta\varphi^6}{63} + \frac{\Delta\varphi^8}{126} - \frac{\Delta\varphi^{10}}{1080} + \mathcal{O}(\Delta\varphi^{12})$
$5/2$	$\frac{5\Delta\varphi^8}{462} - \frac{11\Delta\varphi^{10}}{3024} + \frac{7387\Delta\varphi^{12}}{12972960} + \mathcal{O}(\Delta\varphi^{14})$	$-\frac{5\Delta\varphi^8}{462} + \frac{5\Delta\varphi^{10}}{1386} - \frac{19\Delta\varphi^{12}}{33264} + \mathcal{O}(\Delta\varphi^{14})$
3	$\frac{\Delta\varphi^{10}}{286} - \frac{151\Delta\varphi^{12}}{102960} + \frac{3907\Delta\varphi^{14}}{13366080} + \mathcal{O}(\Delta\varphi^{16})$	$-\frac{\Delta\varphi^{10}}{286} + \frac{5\Delta\varphi^{12}}{3432} - \frac{\Delta\varphi^{14}}{3432} + \mathcal{O}(\Delta\varphi^{16})$





Summary

- All quantum states dual to Bañados (vacuum) geometries saturate QNEC.
- But not all states in CFT2 saturate QNEC.
- QNEC does not saturate when the RT-surface crosses bulk matter.
- For the Vaidya quench this leads to QNEC half saturation of large entangling regions and infinities when the surface enters the matter shell.
- Including finite-c (quantum-gravity) corrections in the bulk requires to compute bulk entanglement and quantum backreacted RT-surfaces.
- Quantum corrections induce small deviations from QNEC saturation.

QNEC from semi-classical Gravity

[Bousso-Fisher-Koeller-Leichenauer-Wall 1506.02669]

Generalized second law $dS_{gen} \geq 0$ requires introduction of generalized entropy

$$S_{gen} = S + \frac{A}{4\hbar G_N}$$

Generalized entropy for general surface allows to define quantum expansion Θ

$$\theta = \frac{1}{\mathcal{A}} \frac{d\mathcal{A}}{d\lambda} \qquad \mathcal{A} \to 4 G_N \hbar S_{gen} \qquad \Theta = \theta + \frac{4 G_N \hbar}{\mathcal{A}} S'$$

Allows to uplifted the focusing theorem to the semi-classical Quantum Focusing Conjecture

$$\begin{aligned} 0 \geq \Theta' &= \theta' + \frac{4 G_N \hbar}{\mathcal{A}} (S'' - \theta S') \\ &= -\frac{1}{2}\theta^2 - \sigma_{ab}\sigma^{ab} - 8\pi G_N \langle T_{kk} \rangle + \frac{4 G_N \hbar}{\mathcal{A}} (S'' - S'\theta) \end{aligned}$$

For vanishing shear and classical expansion this gives QNEC

$$\cancel{G_N} \langle T_{kk} \rangle \geq \frac{\cancel{G_N} \hbar}{\mathcal{A}} S''$$

Example: Steady State Formation